HYPERBOLIC FUNCTIONS ON SLIDE RULES

and

CIVIL ENGINEERING APPLICATIONS

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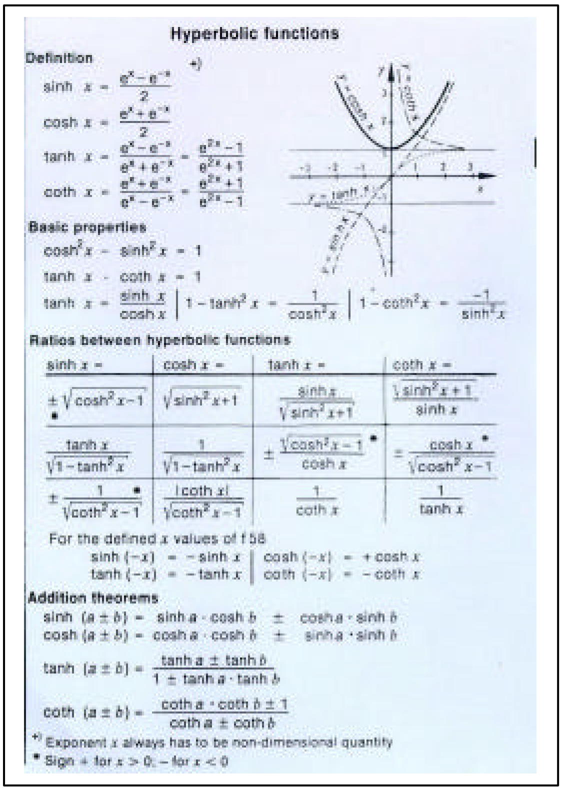
1. PURPOSE OF THIS ARTICLE

Having read the excellent article "DE KETTINGLIJN EN HYPERBOLISCHE FUNCTIES" presented by Simon van der Salm in the October 2000 MIR 26 edition, I investigated, more carefully, my own slide rules collection in order to find out which one are bearing one or more "hyperbolic scales".

The present article is the result of this investigation and of related subjects. Its purpose is to bring some complementary contribution to Simon's article with another lighting.

2. THE HYPERBOLIC FUNCTIONS

First of all, for the friends collectors who are hooked to the mathematical background, in addition to the formulas already supplied by Simon, here below some general mathematical ratios and theorems relating to these sophisticated functions.



Hyperbolic functions

Definition *)

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$

$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}$

Basic properties

$\cosh^2 x - \sinh^2 x = 1$

$\tanh x \cdot \coth x = 1$

$\tanh x = \frac{\sinh x}{\cosh x}$ | $1 - \tanh^2 x = \frac{1}{\cosh^2 x}$ | $1 - \coth^2 x = \frac{-1}{\sinh^2 x}$

Ratios between hyperbolic functions

$\sinh x =$	$\cosh x =$	$\tanh x =$	$\coth x =$
$\pm\sqrt{\cosh^2 x - 1}$ •	$\sqrt{\sinh^2 x + 1}$	$\frac{\sinh x}{\sqrt{\sinh^2 x + 1}}$	$\frac{\sqrt{\sinh^2 x + 1}}{\sinh x}$
$\frac{\tanh x}{\sqrt{1 - \tanh^2 x}}$	$\frac{1}{\sqrt{1 - \tanh^2 x}}$	$\pm\frac{\sqrt{\cosh^2 x - 1}}{\cosh x}$ •	$\pm\frac{\cosh x}{\sqrt{\cosh^2 x - 1}}$ •
$\pm\frac{1}{\sqrt{\coth^2 x - 1}}$ •	$\frac{\lvert\coth x\rvert}{\sqrt{\coth^2 x - 1}}$	$\frac{1}{\coth x}$	$\frac{1}{\tanh x}$

For the defined x values of f 58

$\sinh(-x) = -\sinh x$ | $\cosh(-x) = +\cosh x$

$\tanh(-x) = -\tanh x$ | $\coth(-x) = -\coth x$

Addition theorems

$\sinh(a \pm b) = \sinh a \cdot \cosh b \pm \cosh a \cdot \sinh b$

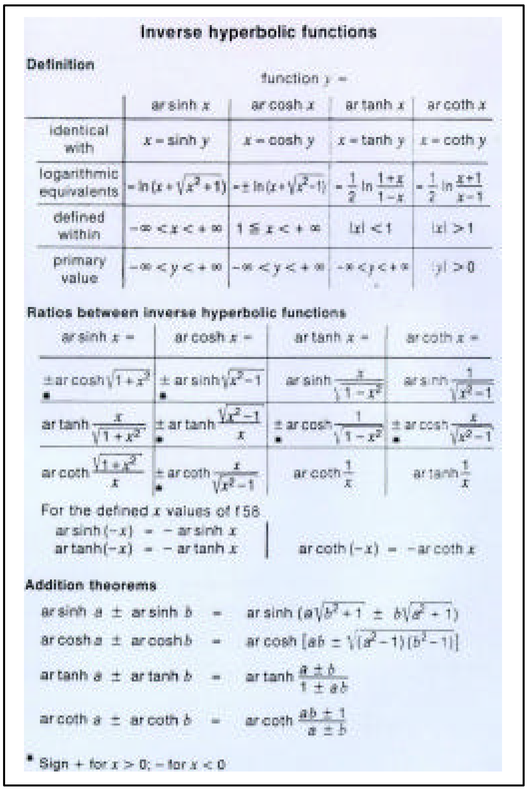
$\cosh(a \pm b) = \cosh a \cdot \cosh b \pm \sinh a \cdot \sinh b$

$\tanh(a \pm b) = \frac{\tanh a \pm \tanh b}{1 \pm \tanh a \cdot \tanh b}$

$\coth(a \pm b) = \frac{\coth a \cdot \coth b \pm 1}{\coth a \pm \coth b}$

*) Exponent x always has to be non-dimensional quantity

• Sign + for $x > 0$; − for $x < 0$



Inverse hyperbolic functions

Definition

function y =

	ar sinh x	ar cosh x	ar tanh x	ar coth x
identical with	$x = \sinh y$	$x = \cosh y$	$x = \tanh y$	$x = \coth y$
logarithmic equivalents	$= \ln(x + \sqrt{x^2 + 1})$	$= \pm\ln(x + \sqrt{x^2 - 1})$	$= \frac{1}{2}\ln\frac{1+x}{1-x}$	$= \frac{1}{2}\ln\frac{x+1}{x-1}$
defined within	$-\infty < x < +\infty$	$1 \leq x < +\infty$	$\lvert x\rvert < 1$	$\lvert x\rvert > 1$
primary value	$-\infty < y < +\infty$	$-\infty < y < +\infty$	$-\infty < y < +\infty$	$\lvert y\rvert > 0$

Ratios between inverse hyperbolic functions

ar sinh x =	ar cosh x =	ar tanh x =	ar coth x =
$\pm$ ar cosh $\sqrt{1 + x^2}$ •	$\pm$ ar sinh $\sqrt{x^2 - 1}$ •	ar sinh $\frac{x}{\sqrt{1 - x^2}}$	ar sinh $\frac{1}{\sqrt{x^2 - 1}}$
ar tanh $\frac{x}{\sqrt{1 + x^2}}$	$\pm$ ar tanh $\frac{\sqrt{x^2 - 1}}{x}$ •	$\pm$ ar cosh $\frac{1}{\sqrt{1 - x^2}}$ •	$\pm$ ar cosh $\frac{x}{\sqrt{x^2 - 1}}$ •
ar coth $\frac{\sqrt{1 + x^2}}{x}$	$\pm$ ar coth $\frac{x}{\sqrt{x^2 - 1}}$ •	ar coth $\frac{1}{x}$	ar tanh $\frac{1}{x}$

For the defined x values of f 58

ar sinh$(-x) = -$ ar sinh x

ar tanh$(-x) = -$ ar tanh x | ar coth$(-x) = -$ar coth x

Addition theorems

ar sinh $a \pm$ ar sinh b = ar sinh $(a\sqrt{b^2 + 1} \pm b\sqrt{a^2 + 1})$

ar cosh $a \pm$ ar cosh b = ar cosh $[ab \pm \sqrt{(a^2 - 1)(b^2 - 1)}]$

ar tanh $a \pm$ ar tanh b = ar tanh $\frac{a \pm b}{1 \pm ab}$

ar coth $a \pm$ ar coth b = ar coth $\frac{ab \pm 1}{a \pm b}$

• Sign + for $x > 0$; − for $x < 0$

3. THE HYPERBOLIC SCALES AND SLIDE RULES WITH HYPERBOLIC SCALES

THE AVAILABLE FUNCTIONS

As far as I know, the (only) six existing scales relating to hyperbolic functions are:

1. **sinh (x)** in the range [sinh (x) = 0.1→1.0]

2. **sinh (x)** in the range [sinh (x) = 1.0→10.0]

3. **cosh (x)** in the range [cosh (x) = 1.0→10.0]

4. **tanh (x)** in the range [tanh (x) = 0.1→ 1.0]

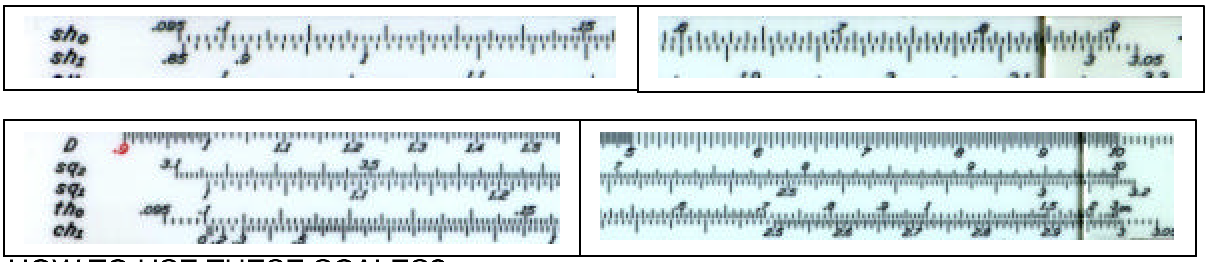
5. **sinh (x) / (x)** in the range [sinh (x) / (x) = 0.1→1.0]

6. **tanh (x) / (x)** in the range [tanh (x) /(x) = 1.0→0.762]

NB. Some slide rules have hyperbolic scales which are not covering the whole range (less or more); see § 0

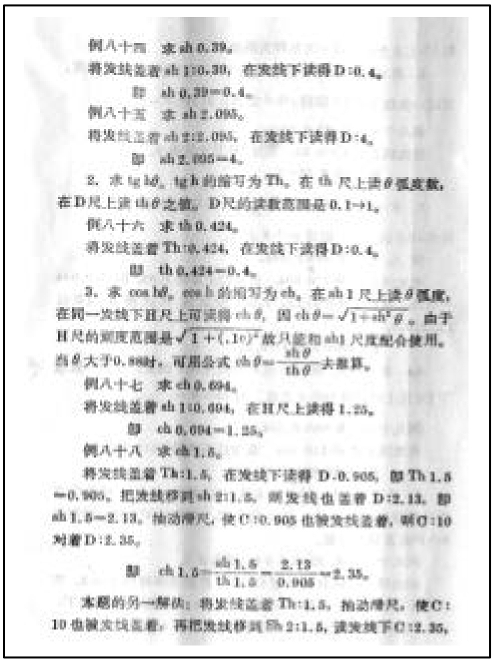
THE SCALES

Here below a partial view of the scales as extracted from the most representative (for hyperbolic functions) slide rule of my collection; the FLYING-FISH 1003.



HOW TO USE THESE SCALES?

For the erudite collectors, I supply here an abstract of the user's manual for the Flying Fish 1015 model ....



例八十四　求 sh 0.39。

将发线盖着 sh 1:0.39，在发线下读得 D:0.4。

即　sh 0.39=0.4。

例八十五　求 sh 2.095。

将发线盖着 sh 2:2.095，在发线下读得 D:4。

即　sh 2.095=4。

2. 求 tg hθ。tg h 的缩写为 Th。在 th 尺上读 θ 弧度数，在 D 尺上读 th θ 之值。D 尺的读数范围是 0.1→1。

例八十六　求 th 0.424。

将发线盖着 Th:0.424，在发线下读得 D:0.4。

即　th 0.424=0.4。

3. 求 cos hθ。cos h 的缩写为 ch。在 sh 1 尺上读 θ 弧度，在同一发线下 H 尺上可读得 ch θ，因 $ch\,\theta=\sqrt{1+sh^2\theta}$。由于 H 尺的刻度范围是 $\sqrt{1+(.1)^2}$ 故只能和 sh1 尺度配合使用。当 θ 大于 0.88 时，可用公式 $ch\,\theta=\frac{sh\,\theta}{th\,\theta}$ 去推算。

例八十七　求 ch 0.694。

将发线盖着 sh 1:0.694，在 H 尺上读得 1.25。

即　ch 0.694=1.25。

例八十八　求 ch 1.5。

将发线盖着 Th:1.5，在发线下读得 D:0.905，即 Th 1.5=0.905。把发线移到 sh 2:1.5，则发线也盖着 D:2.13，即 sh 1.5=2.13。抽动滑尺，使 C:0.905 也被发线盖着，则 C:10 对着 D:2.35。

即　$ch\,1.5=\frac{sh\,1.5}{th\,1.5}=\frac{2.13}{0.905}=2.35$。

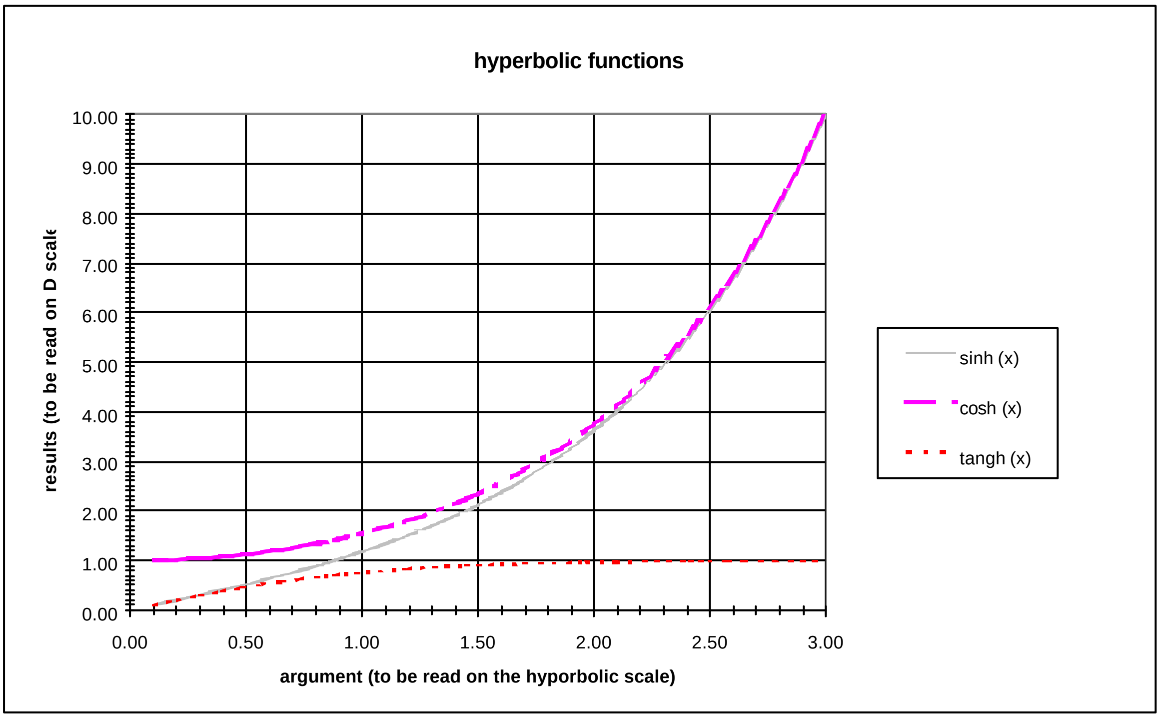
本题的另一解法：将发线盖着 Th:1.5，抽动滑尺，使 C:10 也被发线盖着。再把发线移到 Sh 2:1.5，读发线下 C:2.35。

......for those who have some difficulties in following that text, let us say that the use of the scales is the same as for a conventional trigonometric function : for instance reading "0.80" on scale "th0" and finding "0.66" on scale "D" **[tanh (0.80)** = 0.6640].

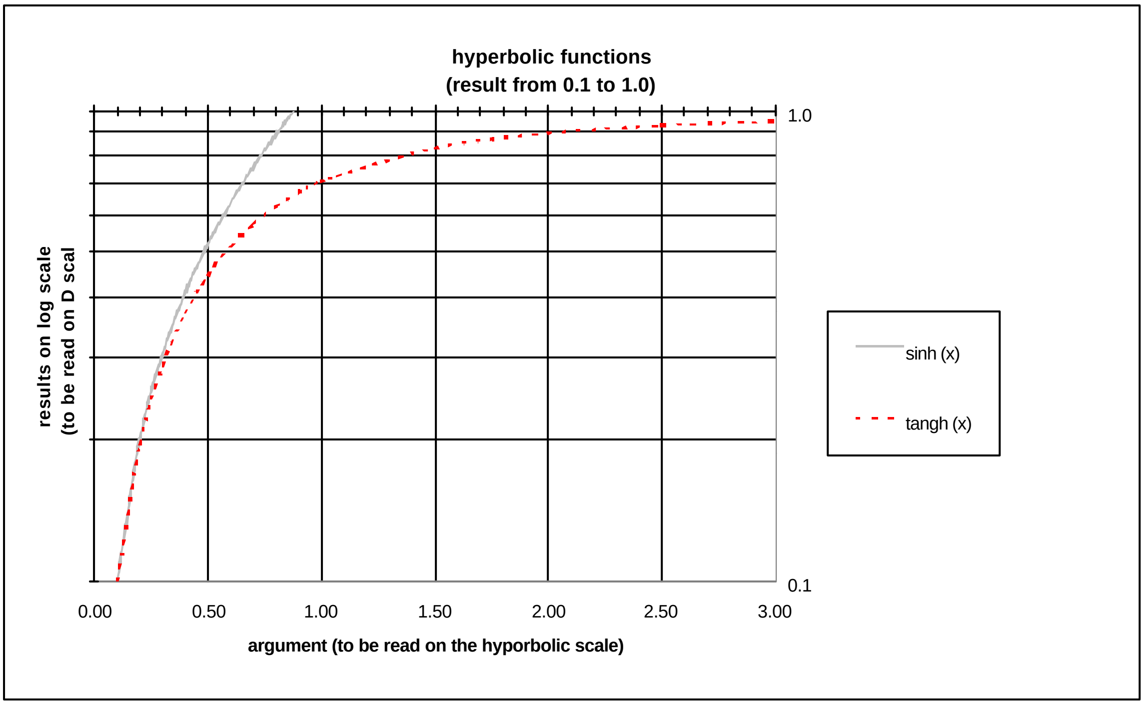
THE FUNCTIONS IN THE RANGE OF THE SLIDE RULES SCALES

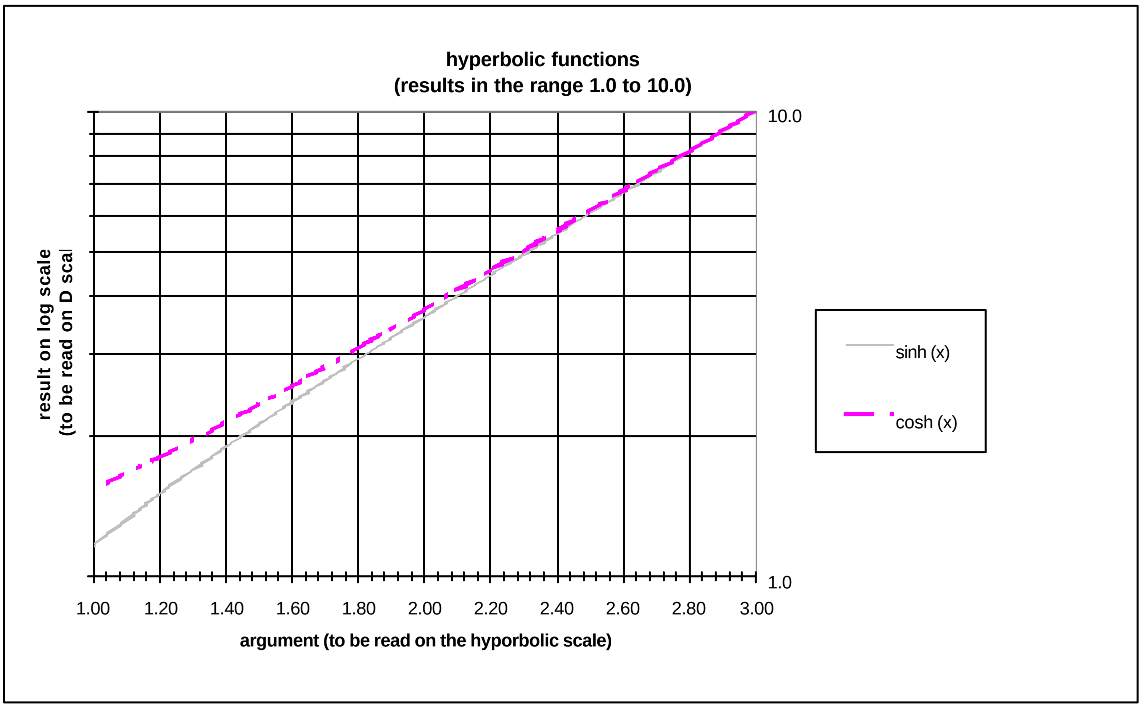
The hyperbolic functions are, on all slide rules, limited by a results range extending on two cycles only (from 0.1 to 1.0 and 1.0 to 10.0), with some exceptions as mentioned in § 0.

The following graph, built on decimal scales, is showing graphically the functions in that range.



The following two graphs are showing the same functions but with the result given on a logarithmic scale which is more in accordance with the way the slide rules scales "D" are built.





Comments

1) For arguments in the range of 2.50 to 3.00 (the results ranging from 6 to 10) the results of the **cosh (x)** and **sinh (x)** are very close (at least by reading on a slide rule); this could be an explanation why the scale **cosh (x)** is rather rare [the **cosh (x)** could nevertheless be obtained indirectly by the equation "**cosh^2(x) = 1 + sinh^2(x)**" or "**cosh (x) = sinh (x) /tanh(x)**"]

2) The **cosh (x)** and **sinh (x)** functions scale are nearly linear on a semi-logarithmic. scale diagram; the D scale is thus well appropriate for such functions.

THE SLIDE RULES

Looking in my collection, I found the following slide rules with the following hyperbolic scales (the symbolism of the scales are reproduced as shown on the slide rules)

Brand	type	Country	Hyperbolic scales				comment
			sinh (x) 0.1 →1.0	sinh (x) 1.0 →10.0	cosh (x) 1.0 →10.0	tanh (x) 0.1 →1.0	
Aristo	0971 hyperbolog	Germany	Sh1	Sh2		Th	
Ding feng	--	China	Sh1	Sh2		Th	
Ding feng	5471	China	Sh1	Sh2		Th	
Ding Feng	5852	China	Sh1	Sh2		Th	
Faber-Castell	2/84 mathema	Germany	sinh x^g	0.1 sinh x^g	0.1 cosh x^g	tanh x^g	**note (1)**

Brand	type	Country	Hyperbolic scales				comment
			sinh (x) 0.1 →1.0	sinh (x) 1.0 →10.0	cosh (x) 1.0 →10.0	tanh (x) 0.1 →1.0	
Flying Fish	1002	China	sh2	sh3		th2	
Flying Fish	1003	China	sh0	sh1	ch1	th0	
Flying Fish	1004	China	Sh1	Sh2	Ch	Th	
Flying Fish	1015	China	Sh1	Sh2		Th	
Haiou Pai	6531	China	Sh1	Sh2		Th	
Haiou Pai	6631	China	Sh1	Sh2		Th	
Hua Fang	--	China	Sh1	Sh2		Th	
Jang	PAT 129	China	Sh (θ)/(θ) **note (2)**	SH **note (2)**		TH **note (2)**	Th(θ)/(θ) **note (2)**
K & E	4083-3	USA	Sh1	Sh2		Th	
Pickett	N 4-ES	USA	Sh1	Sh2		TH	
Sida	1002	China	sh2	sh3		th2	
Sida	1012	China	sh0x	sh1x	ch1x	th0x	
Sida	1015-1	China	Sh1	Sh2		Th	
Sida	1015	China	Sh1	Sh2		Th	
Sida	1083	China	Sh1	Sh2		Th	
Sida	6171	China	sh2	sh3		th2	
Sida	6201	China	sh1	sh2		Th	
Sun Hemmi	255D	Japan	Sh2	Sh1		Th	
Xuesh	6171 bachelor	China	sh1	sh2		th	
Xuesh	6171	China	sh1	sh2		th	

Comments:

1) The above table contents 25 different slide rules; most of them are from China (apparently a country where the hyperbolic functions are rather popular !).
2) Only 4 of them are bearing a **cosh** scale
3) The labelling symbolism is far to be standard, even within the same brand.

4) The two **sinh (x)** scales of the Pickett are located on both sides of a single line (dual-base system).

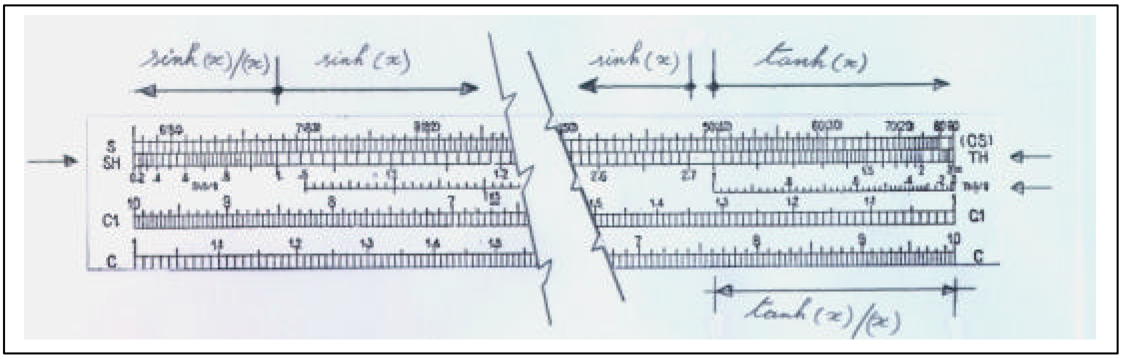
<u>Notes</u>

(1) The Faber-Castell arguments are to be expressed in grads (1/400 th of the circle). This generally induces a boring complication. Furthermore, the basic scale C (labelled "y" on the FC 2/84) having a physical length of 200 mm, is largely extended on both sides. This leads to results ranging from 0.065 to 15 (arguments of **sinh (x)** ranging from 0.065 up to 3.40), which is more than on any other slide rule.

(2) The Jang bears 4 scales but it looks like two scales only. The reason is that one scale (developped on the full length of the 1 cycle C scale), is in fact the juxtaposition of three different scales: with a discontinuity at 2.7.

- **Sinh (x) / (x)** for (x) lower than 1.0,
- **Sinh (x)** for (x) between 1.0 and 2.7
- **Tanh (x)** for (x) between 1.0 and ∞ (close to 3)

Another scale is dedicated to **tanh (x) / (x)** for (x) between 1.0 and ≈ 0.



4. <u>SOME TYPICAL ENGINEERING PROBLEMS WHERE USING THE HYPERBOLIC FUNCTIONS</u>

In addition to the well known chain (cable) case, they are in the field of the civil and/or mechanical engineering, several others interesting cases where the hyperbolic functions are playing a major role.

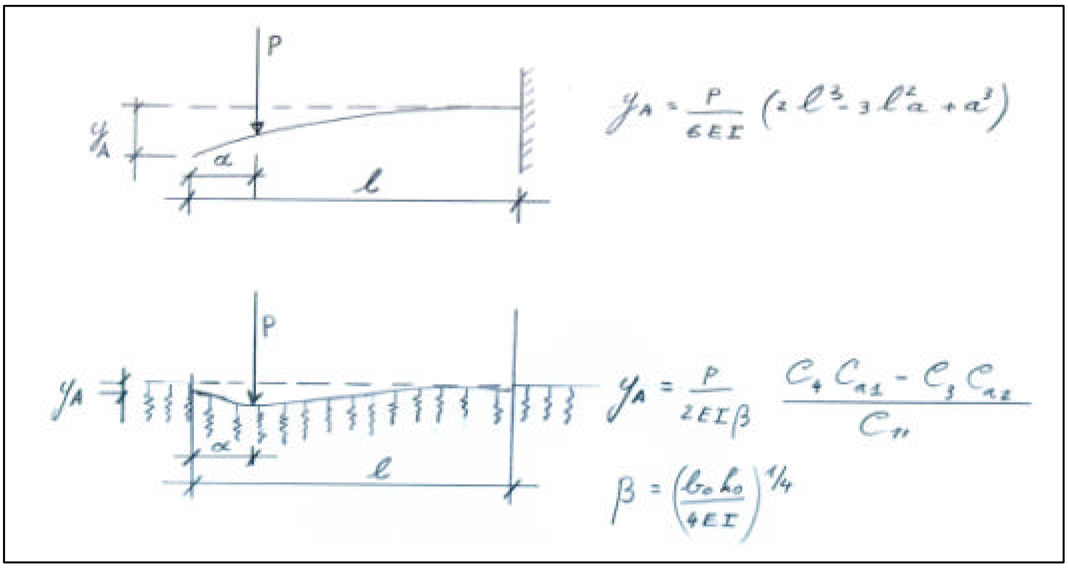
BEAMS ON ELASTIC SOIL (FOUNDATIONS)

Comparing a standard beam problem (as a cantilever beam) with a beam on elastic soil, the difference lies mainly in the supports which are discrete and stiff in the first case and nearly continuous and elastic in the second case. This makes the solution of such a problem much more complicated and requires the use of a huge amount of intermediate calculations involving all kinds of hyperbolic functions.

The example shown below, comparing a standard beam with one on elastic soil, asks, to just find out the deflection at the origin, the computation of the following intermediate data:

- C_4 = **cosh (bl)** * sin (βl) -**sinh (bl)** * cos (βl)
- C_{a1} = **cosh (b(l-a))** * cos (β(l-a))
- C_3 = **sinh (bl)** *sin (βl)

- C_{a2} = **cosh (b(l - a))** * sin (β(l-a)) + **sinh (b(l - a))** * cos (β(l-a))
- C_{11} = **sinh^2 (bl)** - sin^2 (βl)
- The beam's parameters b, E, & I relate to the width, elasticity modulus and moment of inertia.
- The soil's parameter k relates to the soil reaction modulus (expressed in N/mm^3).

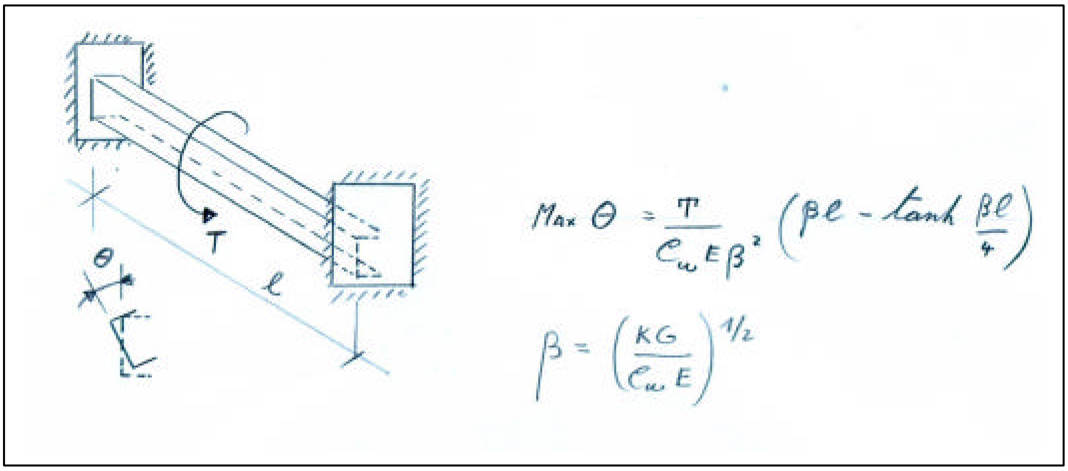


TORSION

Torsion stress calculation is a rather simple problem for a circular bar loaded by equal and opposite twisting couples, applied at its ends.

When the bar is not circular (as a for a U shape section) and when one or both ends are fixed, the formulas are much more sophisticated.

The example given here below deals with a channel bar fully fixed at both ends (no twist or wrap allowed).

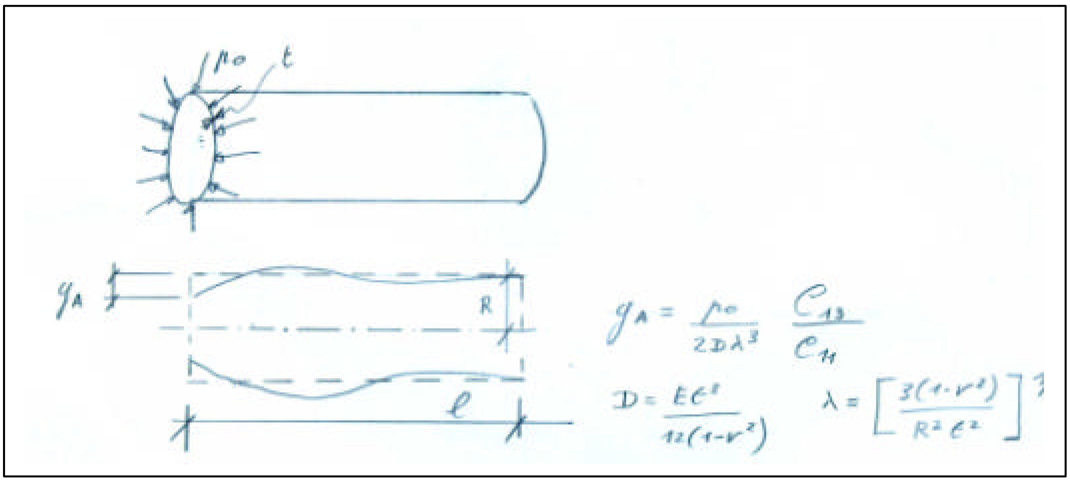


THIN WALLED CYLINDRICAL SHELLS (TUBE)

Even for axisymmetric loadings the equations are not simple. They are based on differential equations similar in form to those used to develop the formulas for beams on elastic soil.

The example given here below deals with a tube loaded by an uniform distributed radial end load. The deflection determination at the end asks for the computation of the following intermediate data:

- C_{11} = **sinh^2 (l l)** - sin^2 (λl)
- C_{13} = **cosh (l l)** * **sinh (l l)** - cos (λl) * sin (λl)
- The parameters ν and E are the Poisson's coefficient and the elasticity modulus.



WATER FLOW IN THE GROUND

A typical problem of water flow in the ground is the one of a sheetpiling wall *[palplanches - damwand]* retaining both the ground and the water acting on and in the retained ground. The computation of the water flow passing through the soil and filling the excavation is a function of the geometry of the system as well as of the soil layers geometry and permeability characteristics.

The flow per linear meter is computed by the formula q = (2kH/π) * ψ_b

where ψ_b has to be determined by the formula b/d = -1/π *(**sinh y** $_b$ /sin φ_c + **y** $_b$**)**

φ_c has to be determined by the formula c/d = -1/π *(cotg φ_c + φ_c)

k is the soil permeability (expressed in m/sec)