## Graphic Logarithmic Tables ${ }^{1}$

## A Picture Should Be Worth A Million Numbers ${ }^{2}$



Given what they replaced it feels disparaging, impolite and almost blasphemous to point out that logarithmic tables were sadly error prone and irritatingly awkward to use. Surely graphic logarithmic tables must have been the answer!


It is fitting to remember in 2014, the year of their 400th anniversary, that before John Napier (1550-1617) invented logarithms most forms of computation were complex and enormously time-consuming. In fields like navigation and astronomy it took elite mathematicians of the day ${ }^{3}$ literally years to finish some calculations. So what on earth could there be about logarithmic tables to complain about?

## Catch-22

The work on logarithms undertaken by Napier was a classic Catch-22 style paradox ${ }^{4}$. First he had to work through the tedious calculations needed to create all the entries in a set of logarithmic tables and then find a way to verify them. But the supreme irony is that once the tables existed, they would have made it much easier to do the calculations in the first place! A modernday Catch-22 analogy would be computer programming. Once a program compiler was developed, programmers no longer had to write programs in machine code. But first someone had to write a compiler in machine code.


Napier, aided by his own "Napier bones", did set about the enormity of the task with considerable insight and ingenuity. His first challenge was to shake off $16^{\text {th }}$ century mathematical thinking of the time.

[^0]Before Napier invented logarithmic tables mathematicians of the day relied on fixed sequences. For example, the fixed arithmetic sequence of $0,1,2,3,4$, etc or the fixed geometric sequence of $1,2,4,8$, 16 , etc. This worked fine when stepping through a series of whole numbers but clearly overlooked all the values in between. So Napier broke with tradition and decided to adopt a kinematic approach for building his tables. This way he would have "no numerical gaps" - i.e. he defined: (i) an arithmetic and continuous movement of a point along a single straight line with a constant speed AND (ii) a proportional and continuous movement of point along a straight line with a proportional speed.

The completeness of his tables was not the only "Catch-22" dilemma Napier had to overcome. He had no precedent for the degree of accuracy the values in his tables needed to have. With this quandary came another conundrum - for a given degree of accuracy, how much work would that involve to generate the full table? So Napier came up with what today would be called a meta process. Having developed a means of calculating a series of values, he would then evaluate the values obtained to see how they could be used to generate more values i.e. a meta calculation process.

So by adopting a kinematic approach and a meta process, Napier achieved an insightful balance between the desired degree of accuracy and the tedious longhand calculations needed to generate the table entries.


Part of Napier's "Mirifici Logarithmorum Canonis Descriptio"

## Published errors

Inevitably a few errors did creep into Napier's original calculations and indeed into the calculations made by later authors who devised extended versions (e.g. greater number of significant places) or designed new types of logarithmic tables. Some of the earliest errors went unnoticed and were repeated or compounded in some of the tables published much later. But even if it had been humanly possible for Napier to generate and note down all the entries needed for his logarithmic table totally error-free, many mistakes got introduced during the typesetting and printing.

When copying or duplicating a long list of numerals, apart from any other oversights, unintentionally transposing two adjacent numbers is a known human weakness. Then came the thankless task poorly paid typesetters had to face. Typesetting row upon row of seemingly random numerals in a printing block must have been a really mind-numbing task. It was unintentional but understandable how typesetting errors ended up in many logarithmic tables. For example, errors made in the last decimal place
were particularly difficult to spot. Such mistakes often remained unnoticed for decades and were perpetuated when inherited from one publishing house to another and logarithmic tables got reissued or republished.

## Logarithmic tables can be a pain

Arguably, given the longhand nature of calculating the table entries and then the mind-numbing typesetting needed to publish them, the odd error was a small price to pay for how logarithmic tables made all kinds of multiplication and division much simpler and momentously quicker to perform. As it turned out, the immeasurable gains that logarithmic tables made possible far outweighed the more obvious tangible benefits. Top of the list of immeasurable gains were:

1. individuals other than "mathematical geniuses" could now attempt complicated calculations
2. scientists and mathematicians could now (in their lifetimes) unlock and solve many mathematical conundrums which in turn lead to many advancements and discoveries in many different fields

However, despite being a paradigm shift when compared with the old pre-logarithm ways, the very nature of the concept meant using tables had distinct disadvantages. The ritualistic look-up process could be irritating and so long-winded that without fastidious care, it could itself easily became error-prone. Also some factors commonly found in calculations could make using early logarithmic tables tricky and a real pain to use. For example, negative numbers or answers needing a high degree of accuracy - say significant to at least 7 or 8 significant places. A workaround for negative numbers was to handle the sign separately. Often the drawbacks got magnified many times over when attempting complex calculations. Finally the sheer number of times an interim solution had to be looked up in a logarithmic table became a test of concentration and patience.

So despite the many obvious advantages, logarithmic tables were not error-free and using them was error-prone and irritating. But centuries after logarithmic tables were first published an innovative and elegant solution was found most of the drawbacks and the tedium - graphic logarithmic tables.

## Nomograms showed the way

Like many great ideas, the appeal of a nomogram comes from its simplicity. It is a two-dimensional diagram designed to show the approximate "graphic calculation" of a mathematical function. The most basic nomogram having two parallel outer scales representing the values of two quantities involved in a function. Where the lines joining quantities used in the calculation intersect, gives the result of the function. An early advocate of the nomogram was Frenchman Leon Louis Lalanne (1812-1892). In 1843, with his "Universal Calculator", he probably created the first log-log plot.

Sadly some nomograms became so complex that the function(s) they represented were just as difficult to grasp as their inherent longhand mathematical formula. However, it is the same

basic idea that a picture is easier to understand than a long list of numbers that led to graphic logarithmic tables being developed.

## So what are graphic logarithmic tables?

Printed as a single linear scale (end-to-end scale lengths varying from as little as 4.1 m to as much as 115 m - see Appendix $A$ for the details) they clearly look strikingly different from the many rows and columns of similar looking numerical entries in a traditional logarithmic table.


The beginning and the end of an example graphic logarithmic table

Instead they somewhat reflect the concepts of a calculation aid based on rods and scales Professor Johann Gottfried Steinhäuser theorised about in 1807. A closer match is the intriguing prototype disc for a graphic logarithmic table ${ }^{5}$ proposed by a medical officer in the French Army, Dr Haro, in 1887. Regardless of the inspiration, the differences when compared to traditional logarithmic tables are much more than just an innovative printing or formatting style.

In a graphic logarithmic table:

- all the entries in the log and antilog sections of a traditional logarithmic table have been integrated into a single, much shorter "endless" entry
- the look-up process is simpler, quicker and much more intuitive

But it is also important to set boundaries on what constitutes a graphic logarithmic table as they come in various shapes and sizes and many loosely related "cousins" exist. For example, a slide rule (in its many forms) has logarithmic scales that provide answers without needing to resort to antilogs. However, for practical reasons the logarithmic scales normally found on slide rules only ever represent a subset of all the entries found in any traditional or graphic logarithmic table. By using a subset or leaving out rarely needed parts of the range was one of the few ways slide rule designers could usefully compact the length of a scale and ultimately, the length of the slide rule. Only the largest drum and grid-iron types had enough room for a range of values that got close to the full range of values included in any logarithmic table. Without these boundaries, a plethora of other printed calculating aids and slide charts could arguably have been called types of graphic table.

So unlike slide rules and similar calculating devices, graphic logarithmic tables still rely on a table look-up process and apart from one exception, all known examples exist as some form of printed page or more commonly, a slim book.

[^1]
## Graphic logarithmic tables - how did they work?

The easiest way to show how such graphic logarithmic tables were used is a worked example. However, the developers of such tables chose an eclectic variety of ways to achieve the same goal. Despite these design differences, the way they were used, when compared with a traditional logarithmic table, is universal.

The chosen example, $2.5 \times 5$, looks trivial but enough to show the generic process. The only drawback is that the simplicity of the example hides the full tedium of and error-prone nature of the repetitive look-up process when using traditional logarithmic tables for complex calculations. Conversely the advantages of a graphic logarithmic table are amplified many times over for such complex calculations.

Using a traditional table of logarithms
With a table compiled for 5 decimal places ${ }^{6}$ the minimum calculation steps are:

|  | $=$ |
| :--- | :---: |
| 1. Look-up the logarithm of 2.5 | 0.39794 |
| 2. Look-up the logarithm of 5 | 0.69897 |
| 3. Add the Log of 5 to the Log of 2.5 | 1.09691 |
| 4. Look-up the antilog of the mantissa 09691 | 12500 |
| 5. Use the characteristic "1" before the mantissa to <br> fix the decimal point | $\mathbf{1 2 . 5}$ |

Depending on the notation form/style of the entries (especially the antilog entries) in a traditional table each look-up step in more complex calculations could well have required extra interim interpolation steps to determine the logarithm of each number and the antilog of the resulting mantissa.

Using the graphic logarithmic table by Lacroix and Ragot
Opting for the longer 40-page table for five decimal places in the book by Lacroix and Ragot (see Appendix $A$ for the details) the logarithm of 2.5 can be quickly and easily found.


[^2]Ignoring the decimal point and looking up the leading " $\mathbf{2 5}$ " digits in the " $\mathbf{N}$ " column is enough to find the right page in the table. In the adjacent column "L" the leading digits of the logarithm, "39", are shown. The next step is to locate the following two " $\mathbf{0 0}$ " digits of the number on the upper scale of graduations for $\log$ section " 39 " of the table. The $3^{\text {rd }}$ and 4 th digits, " $\mathbf{7 9}$ ", of the logarithm can found on the lower scale to the left of the tick mark " $\mathbf{0 0}$ ". Finally counting the extra divisions/tick marks that come after 79 before lining up with 00 on the upper scale, gives the last digit of the logarithm: " 4 ". So, the complete readout is $39|\mid 79 \| 4$ or $\log 2.5=0.39794$.

The logarithm of " 5 " can be found equally easily. Adding the two logs (39794 and 69897) gives the same 1.09691 interim answer. But as the log and antilog entries are combined in a graphic logarithmic table, the antilog of the mantissa, "09691", can be quickly and easily "reverse engineered" using the same intuitive process.

This time looking-up leading "09" digits of the mantissa in column "L" are enough to find the right page in the table and from the column alongside, read off the leading digits of the answer: "12". Having found the next two digits, " $\mathbf{6 9}$ ", in the log " 09 " section, one tick mark further for the trailing " 1 " in the mantissa and the corresponding last two digits of the antilog number can be read off the upper scale i.e. " $\mathbf{5 0}$ ". After concatenating the two parts and the using the characteristic of the mantissa to fix the decimal point, $\mathbf{1 2 . 5}$ is the answer.

Superficially the steps look similar to using a traditional table of logarithms. However, the graphic version, with its fewer pages and combined log and antilog entries, is certainly less error-prone and much more intuitive to use. Also although both types of tables are compiled for five decimal places, only the graphic version has the inherent potential for accuracy to six decimal places. The values in the

worked example finish exactly on a "tick mark" in the scales. But when needed and much like using a slide rule, accuracy to a sixth decimal place by interpolating between two tick marks would be simple and easy to achieve with this table.
Using the graphic logarithmic table by Koch and Putsbach

By contrast the table by Koch and Pulsch (see Appendix A for the details) needs interpolation to achieve accuracy to four decimal places. But strikingly the table is just 4 pages long - highlighting the compactness possible with graphic logarithmic tables compiled for a limited number of decimal places. Needless to say the logarithm of 2.5 can be just as easily found with this version.

Ignoring the decimal point, the leading " $\mathbf{2 5}$ " digits of the number is found in the $1^{\text {st }}$ column on the 2 nd page of the table. The next step is again to find the following two " $\mathbf{0 0}$ " digits of the number on the upper scale of graduations for line 25 . The corresponding logarithmic value for " $\mathbf{0 0}$ " on the upper scale is between "3970" and "3980" on the lower scale - in fact nearly but not quite " $\mathbf{3 9 8 0}$ ". Using interpolation for the $4^{\text {th }}$ digit, the full readout is $397 \| 9$ or with this table $\log 2.5=0.3979$.

Again the logarithm of " 5 " can be found equally easily. Although this time, the interim answer after adding the two logs (" 3979 " and " 6989 ") is not unsurprisingly slightly less accurate: $\mathbf{1 . 0 9 6 8}$. As with all such graphic logarithmic tables, the antilog of the mantissa can be just as quickly "reverse engineered" using the same intuitive process already described.


Again the leading " $\mathbf{0 9}$ " digits of the mantissa are used to find the right page in the table and to read off the leading digits of the answer: " $\mathbf{1 2}$ ". The " $\mathbf{0 9 6 8}$ " mantissa lies between two "tick marks": " $\mathbf{0 9 5 0}$ " and " $\mathbf{1 0 0 0}$ ". Using interpolation to fix " $\mathbf{0 9 6 8}$ " on the scale, the corresponding last two digits of the antilog number can be read off the scale at the top of the page - i.e. "49". The characteristic of the mantissa again gives the final answer but this time the cumulative effect of working to fewer decimal places means the answer comes out as less precise: 12.49.

## Graphic logarithmic tables - no panacea

Given my opening "A Picture Should Be Worth A Million Numbers" gambit, graphic logarithmic tables should surely have superseded the tedious use of traditional logarithmic tables. They did not. Instead graphic logarithmic tables are largely unknown and rare.

This could be because schools and educational institutions of the day preferred to stick largely to using conventional (and cheaper) mass-produced books of traditional logarithmic tables. But the more probable explanation is that most examples of graphic logarithmic tables are early 20th century developments and some telling inherent limitations meant they were outdated almost as soon as they were published.

But who were the intended users of graphic logarithmic tables? Clues can be found in the "Introductions" of the more well-known graphic logarithmic tables such as Pressler, Lacroix and Ragot and Le$d e r$. The advantages commonly quoted are speed and size. Reducing the labour-intensive and errorprone process of using a traditional logarithmic table would have obviously appealed to many professions and trades. Condensing the hundreds of pages of a traditional logarithmic table down to a slim volume would also have been preferable to carrying around a bulky book. A modern-day analogy is how the slim iPad is commonly preferred to a bulky laptop computer. Ernst Leder goes on to suggest that graphic logarithmic tables could also be: "a good tool for further education." However, early 20th century students who could have been attracted by the advantages of a graphic logarithmic table would almost certainly have opted instead for one of the superior aids of the day - such as the slide rule. Therefore it is even more surprising that a renowned German slide rule maker decided that selected slide rules models would be sold with a graphic logarithmic table!

## "Selling fridges to Eskimos"

Apart from sharing logarithmic roots, slide rules and graphic logarithmic tables have little in common. They are competing rather than complementary calculating aids. However, German Carl Kübler (18751953) must have been a great salesman.

In 1940 Faber-Castell (F-C) decided to supplement their existing portfolio of slide rules with a series of "combination" models. They uniquely were the first to incorporate a mechanical flat sliding bar adder for addition and subtraction into the back of their more popular models. These hybrids became known as their Addiator slide rules ${ }^{7}$ because F-C bought the metal Troncet-type slide adders with an accompanying stylus from Addiator $G m b H$, a company founded by Carl Kübler in 1920. This made good commercial sense as the company was the leading maker of slide adders or Addiators.


What is less clear is how Kübler persuaded F-C they also needed to buy an accompanying 2-page "Maximator" paper graphic logarithmic table he had earlier copyrighted. It is illogical but for a brief inexplicable initial two-year period (1940-1942) F-C sold their 1/22A (Disponent), 1/54A (Darmstadt) and 1/87A (Rietz) hybrid models with a "Maximator" graphic logarithmic table inserted behind a glued paper strip at the back of the respective instruction booklets (booklet no.'s: 1/702, 1/704 and 1/707). All the booklets included a page on how to use the table.

After 1942 F-C dropped the "Maximator" and opted for a generic instruction booklet for all their Addiator models!

## Final paradox

Ironically having started with a Catch-22 paradox I conclude with another. Early in their evolution compiling and typesetting traditional logarithmic tables was a challenge. Although inherently less error

[^3]prone, in such times that the graphic printing possibilities were extremely crude and virtually non-existent. By contrast, in the $20^{\text {th }}$ century the possibilities for printing complex images and graphically complex figures were bountiful. This meant graphic logarithmic tables were now relatively easy and economical to publish. However, by the $20^{\text {th }}$ century demands for accuracy had risen sharply.

By their very "picture" nature most graphic logarithmic tables, even with the most precise printing or production techniques, only offered 4 or 5 significant places of accuracy. But by now traditional logarithmic tables of 7,8 or many more significant places had been common place for decades.


The "Maximator-Erweiterungs-Tabelle" graphic logarithmic table
Once early $20^{\text {th }}$ century cheaper printing and production techniques became readily available, graphic logarithmic tables could flourish. But sadly by now their level of accuracy had been surpassed and they faced competition from slide rules and other mechanical aids. This meant almost as soon as they became a practical reality, graphic logarithmic tables were outdated and inferior.

So unlike the traditional logarithmic table, graphic logarithmic tables (even the often reprinted Lacroix and Ragot) never fulfilled their promise and never became well-known or widely used.

## Acknowledgements \& Bibliography

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## Appendix A: Known Graphic Logarithmic Tables

Such tables defy inclusion into any existing classification scheme for similar aids such as Ready Reckoners, Tabular Calculators, etc. So in this compendium (others may well exist) of all the known commercially printed examples ${ }^{8}$ are listed, with a thumbnail image, by the year they were first published.

> +/-1852: Pressler - Dresden, Germanv


1893: Loewe - Bromberg, Germany


[^4]
## 1897: Tichy - Wien, Austria

\(\left.\begin{array}{ll}Title: \& Graphische Logarithmen-Tafeln <br>
By: \& Anton Tichy <br>
Type: \& Hardback book (30 pages) with light fawn cover <br>

Size: \& 24.5 \mathrm{~cm} \times 16.0 \mathrm{~cm}\end{array}\right]\)| Published by: |
| :--- |
| Parlag des Oesterr. Ingenieur- und |
| Architekten-Vereines, Wien |$\quad$| None found |
| :--- |
| Style of table(s): | | (i) Four-place graphic table organised in columns |
| :--- |
| (ii) Other tables for trigonometrical functions |



1908: Leder - Berlin, Germany

| Title: | Die Praxis des Logarithmen-Rechnens <br> Ernst Leder <br> Circular cardboard chart with a card- <br> board cursor as part of a hardback book <br> (125 pages) with pale blue linen cover <br> Chart: Ø 21 cm <br> Book: $27.8 \mathrm{~cm} \times 21.9 \mathrm{~cm}$ <br> Verlag der Cito-Rechenmaschinen- <br> Werke G.m.b.H., Berlin |
| :--- | :--- |
| Type: | DE104927 - 15th August 1899 |
| Size: | DE223529 - 24th June 1910 |
| Published by: |  |
| Fatents? | Four-place "graphic table" (antilogs only) organised as radii from the centre of <br> the chart |
| Style of table(s): |  |
| Length(s) of | $\approx 6.1 \mathrm{~m}$ |
| graphic table: | This is the exception to all the other book-style listings. |
| Comments: | Instead of tables, the entire book consists of advice and worked examples of <br> how to use logarithms in a myriad of calculations. A sleeve, pasted onto the in- <br> side back cover, holds a circular graphic antilogarithm chart. Using the chart as <br> an "antilog table" is explained in the book but the author assumes that the |
| looking-up of logarithms (not possible with the chart) is done with a traditional |  |

1925: Lacroix and Ragot - New York, USA

A Graphic Table combining Logarithms and Anti-Logarithms
Adrien Lacroix and Charles L. Ragot Hardback book ( 52 pages) with green linen cover
$23.6 \mathrm{~cm} \times 15.2 \mathrm{~cm}$
The Macmillan Company, New York
US1610706-14 ${ }^{\text {th }}$ December 1926
(i) 40-page five-place without interpolation graphic table organised in rows
(ii) 6-page four-place graphic table organised in rows
Length(s) of Long version $\approx 115 \mathrm{~m}$
graphic table: Short version $\approx 13.8 \mathrm{~m}$


Comments:
Book reprinted in 1927, 1936, 1938, 1941, 1942 and 1943.

## +/-1926: Kübler - Berlin, Germany



## 1946: Kienbaum - Gummersbach, Germany



[^5]
[^0]:    ${ }^{1}$ Research findings by the author published in 2013 as part of the "Collectanea de Logarithmis" DVD.
    ${ }^{2}$ Derived from a $20^{\text {th }}$ century phrase coined by American Frederick R. Barnard.
    ${ }^{3}$ In Napier's day such calculation experts were known as: "Reckoners".
    ${ }^{4}$ The impossible paradox famously introduced in Joseph Heller's 1953 book of the same name.

[^1]:    ${ }^{5}$ As far as is known the concept was never developed further than the paper Dr Haro submitted to the "Association Francaise pour lÁvancement des Sciences" 1887 conference in Toulouse.

[^2]:    ${ }^{6}$ Half of all the traditional logarithm tables ever published were versions for 4 or 5 decimal places.

[^3]:    ${ }^{7}$ Model numbers suffixed by "A" (至ddiator) for 25 cm and by " $R$ " (Addiator) for pocket $121 / 2 \mathrm{~cm}$.

[^4]:    ${ }^{8}$ As defined in the main part of this paper.
    ${ }^{9}$ Nearly two decades later, in 1912, the same Reiss started making slide rules.

[^5]:    ${ }^{10}$ Started a one-man business that later became one of Germany's leading consulting companies.
    ${ }^{11}$ Rohrberg also designed Faber-Castell model 342 Columbus "System Rohrberg" specialist slide rule.

