Mirifici Logarithmorum Canonis

Reconstructio

An exploration of the mathematics of Napier's logarithm

Simon A.M. van der Salm

MIRIFICI LOGARITHMORUM CANONIS CONSTRUCTIO;

Et eorum ad naturales ipsorum numeros habitudines;

VNÀ CVM

Appendice, de aliâ eâque præstantiore Logarithmorum specie condenda.

QVIBVS ACCESSERE

Propositiones ad triangula sphærica faciliore calculo resolvenda: *Unà cum Annotationibus aliquot doctissimi* D. HENRICI BRIGGII, *in eas & memoratam appendicem.*

Authore & Inventore *Ioanne Nepero*, Barone *Merchistonii*, &c. Scoto.

EDINBVRGI,
Excudebat ANDREAS HART.
ANNO DOMINI 1619.

RECONSTRUCTION OF NAPIER'S LOGARITHM TABLE

An exploration of the mathematics of Napier's Mirifici Logarithmorum Canonis *Constructio*

Simon A.M. van der Salm

CHAPTER 1
INTRODUCTION

Abstract

This essay delves into the mathematical foundations of John Napier's groundbreaking work on logarithms, particularly focusing on his seminal text, Mirifici Logarithmorum Canonis Constructio. Through this exploration, we aim to reconstruct Napier's logarithm table and shed light on the innovative techniques and concepts he employed in its creation. Additionally, we will situate this reconstruction within the broader historical context of mathematics in the 16th and 17th centuries.

1.1. Who was John Napier?

John Napier (1550-1617) stands as a monumental figure in the history of mathematics, celebrated primarily for his revolutionary invention of logarithms.

A distinguished Scottish nobleman, astronomer, theologian, and mathematician, Napier was born into a prominent family and held the title of the 8th Laird of Merchiston. This esteemed position afforded him considerable influence in both academic and political spheres in Scotland during his lifetime, further amplifying the impact of his groundbreaking contributions to mathematics.

His development of logarithm tables revolutionized calculations, making them significantly easier and faster, especially in fields such as astronomy, navigation, and surveying. By transforming multiplicative processes into additive ones, Napier's work enabled simpler computation — a crucial advancement prior to the invention of firstly mechanical, then electromechanical, and analog computers, later electronic calculators and digital computers.

Fig. 1.1.1. John Napier (1550 – 1617), 8th Lord of Merchiston near Edinburgh, Scotland. Portrait dated 1616; presented to the University of Edinburgh by his great granddaughter Margaret, who became Baroness Napier in 1686. Artist unknown. Source: National Galleries of Scotland.

His pioneering book, *Mirifici Logarithmorum Canonis Descriptio*, published in 1614, introduced these logarithmic concepts and laid the groundwork for subsequent advancements in computational methods.

In addition to his innovative work on logarithms, he made significant contributions across various fields, including the reintroduction of the decimal point and the advancement of the numeric notation through the decimal place-value system.

Napier also pioneered smart ‘calculating devices’, most famously inventing *Napier's Bones* — a practical calculating tool — and the *Promptuary*, which utilized a combination of various *Napier’s Bones* to streamline complex calculations. These inventions greatly simplified numerical operations and diminished reliance on traditional pen-and-paper methods. For more information, please visit website *7.

Outside his mathematical endeavours, Napier was a deeply religious individual and a passionate scholar who believed that his mathematical discoveries offered profound insights into the nature of creation. He viewed himself primarily as a theologian. In 1593, he published *A Plaine Discovery of the Whole Revelation of Saint John*, a work that received considerable acclaim not only in Calvinist Scotland but also throughout Protestant regions of Europe, which he himself regarded as his greatest achievement. For more information, see Havil (2014), [2], pp. 35-61, and refer to the provided website *1.

1.2. Two books: the Descriptio and the Constructio

In 1614 John Napier published the *Mirifici Logarithmorum Canonis Descriptio*, which includes a table of logarithms for trigonometric quantities. This table provides remarkably accurate numerical approximations of the exact, continuous Napier logarithms (which we will define later and denote by the capital letter L) for the Sine, Cosine, and Tangent (also in capital letters!) of angles in the first quadrant of a circle with a radius of R = 10^7 units. In this configuration, the arc of the quarter circle is divided into 90 x 60 = 5,400 minutes. Refer to Figure 1.2.1 and Table 1.2.1 for further details.

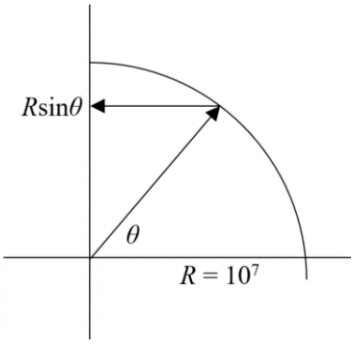

Fig. 1.2.1. The Quarter Circle of Napier. During Napier's era, the term "Sine" (capitalized) referred to what we now recognize as the sine function (lowercase), but with a crucial addition: it was multiplied by the radius R of the relevant circle. Around the year 1600, trigonometric values were primarily viewed as lengths rather than ratios. Napier opted for a radius R of 10^7, which allowed most of his Sine values to maintain seven significant digits to the left of the decimal point, ensuring sufficient accuracy in calculations. For further insights, refer to Macdonald (1889), [1], p. 8.

To achieve sufficient accuracy, Napier aimed to express most of the Sine values in his table — excluding the smallest ones — as integers represented in seven significant figures. To accomplish this, he selected the large radius of R = 10^7. For more information, see Macdonald (1889), [1], p. 8, and Havil (2014), [2], p. 70. The ingenious calculation method used by Napier resulted, after twenty years (sic!) of laborious calculations, in the first published logarithm table for trigonometrical quantities, of which Table 1.2.1 shows the first page.

In 1619, Robert Napier, in collaboration with the London mathematician Henry Briggs[1], posthumously published his father’s earlier work: *Mirifici Logarithmorum Canonis Constructio*. See Napier and Briggs, (1620), [1b]. This publication included notes in which John Napier elucidated the revolutionary analytical foundations of his calculation method. Remarkably, he

[1] Henry Briggs (1561-1630), professor of mathematics at London's Gresham College. Briggs is best known for his work on logarithms with base 10.

had compiled this detailed description several years or maybe more than a decade prior to the publication of the *Descriptio* in 1614, yet he opted not to release it at that time. As a result, the foundation of the *Constructio* likely predates the *Descriptio* by as much as twenty years, even though the latter was published first. For further details, refer to Macdonald (1889), [1], p. xvi.

Thus, although the *Descriptio* was only published in 1614, Napier's invention of logarithms maybe appears to date back to before 1594. There is some additional evidence for that. Kepler, who served as Tycho Brahe's assistant for several years in Prague, mentions in a 1624 letter to the Danziger mathematician Crüger that Brahe received a letter in 1594 from a friend of Napier's in Scotland, suggesting that the publication of the *Mirifici Logarithmorum Canonis* was forthcoming. For more information, see Gravelaar (1899), [14], p. 49.

Deg. 0 +|—

mi	Sines	Logarith	Differen.	Logarith:	Sines	
0	0	Infinite.	Infinite.	.0	1000000.0	60
1	291	8142567	8142568	.1	1000000.0	59
2	582	7449419	7449421	.2	999999.8	58
3	873	7043952	7043956	.4	999999.6	57
4	1164	6756275	6756274	.7	999999.3	56
5	1454	6533131	6533130	1.1	999998.9	55
6	1745	6350810	6350808	1.6	999998.6	54
7	2036	6196659	6196657	2.2	999998.0	53
8	2327	6063128	6063126	2.8	999997.4	52
9	2618	5945345	5945342	3.5	999996.7	51
10	2909	5839986	5839814	4.3	999995.9	50
11	3280	5744676	5744671	5.2	999995.0	49
12	3491	5657665	5657658	6.2	999994.0	48
13	3781	5577622	5577615	7.3	999992.8	47
14	4072	55[illegible]3514	5503506	8.4	999991.7	46
15	4363	5434522	5434513	9.6	999990.5	45
16	4654	5369984	5369973	10.9	999989.2	44
17	4945	5309360	[illegible]309[illegible]48	12.3	999987.8	43
18	5236	5252202	[illegible]252188	13.8	999986.3	42
19	5527	5198136	5198120	15.4	999984.7	41
20	5818	5146843	5146836	17.0	999983.1	40
21	6109	5098054	5098045	18.7	999981.3	39
22	6399	5051534	50[illegible]1514	20.5	999979.[illegible]	38
23	6690	5007083	5007060	22.4	999977.6	37
24	6981	4964524	4964499	24.4	999975.6	36
25	7272	4923703	4923676	26.5	999973.6	35
26	7563	4884483	4884454	28.7	99997[illegible].4	34
27	7854	4846743	4846712	30.9	999969.2	33
28	8145	4810376	4810343	33.2	999966.8	32
29	8436	4775286	4775250	3[illegible]	999964.4	31
30	8726	4741385	4741347	38.1	999961.9	30
					Min.	

Deg. 89

Deg. 0 +|—

mi	Sines	Logarith.	Differen.	Logarith.	Sines	
30	8726	4741385	4741347	38.1	999961.9	30
31	9017	4708596	4708555	40.7	999959.3	29
32	9308	4676848	4676805	43.4	999956.6	28
33	9599	4646077	4646031	46.1	999953.9	27
34	9890	4616225	4616176	48.9	999951.1	26
35	10181	4587239	4587187	51.8	999948.2	25
36	10472	4559069	4559014	54.8	999945.2	24
37	10763	4531671	4531613	57.9	999942.1	23
38	11054	4505004	4504943	61.1	999938.9	22
39	11344	4479030	4478965	64.4	999935.7	21
40	11635	4453713	4453645	67.7	999932.3	20
41	11926	4429022	4428950	71.1	999928.9	19
42	12217	4404925	4404850	74.6	999925.4	18
43	12508	4381396	4381318	78.2	999921.8	17
44	12799	4358408	4358326	81.9	999918.1	16
45	13090	4335936	4335850	85.7	999914.3	15
46	13380	4313958	4313868	89.6	999910.5	14
47	13671	4292453	4292360	93.5	999906.5	13
48	13962	4271401	4271304	97.5	999902.5	12
49	14253	4250783	4250682	101.6	999898.4	11
50	14544	4230583	4230477	105.8	999894.2	10
51	14835	4210781	4210671	110.1	999890.0	9
52	15126	4191364	4191250	114.5	999885.6	8
53	15416	4172317	4172198	118.9	999881.1	7
54	15707	4153627	4153504	123.4	999876.6	6
55	15998	4135279	4135151	128.0	999872.0	5
56	16289	4117263	4117130	132.7	999867.3	4
57	16580	4100664	4100527	137.5	999862.5	3
58	16871	4082175	4082032	142.4	999857.7	2
59	17162	4065082	4064935	147.3	999852.7	1
60	17452	4048276	4048124	152.3	999847.7	0
					Min.	

Deg. 89

Table. 1.2.1. First page of Napier's logarithm table from 1614. In the left most column, divided over two pages, the angles run from 0°0' to 0°60' = 1° from top to bottom. In the right most column the complementary angles run from 89°0' to 89°60' = 90° from bottom to top. So, one reads the left Sinus column from top to bottom; the right Sinus column from bottom to top.

In this essay, we will reconstruct the logarithm table from the *Descriptio*[2] using modern mathematical and computational tools. Our starting point will be the still-readable English version of the *Constructio*, titled *The Construction of the Wonderful Canon of Logarithms*, pub-

[2] In 1614 the *Descriptio* was published as a quarto book of 90 Tables and 57 pages with descriptions of how to use the logarithms, but without an explanation of how Napier calculated them.

lished in 1889. This version is a translation of Napier's original Latin edition from 1614 by William Rae Macdonald. See Macdonald (1889), [1][3].

1.3. Napier's biography by Julian Havil

We would like to acknowledge the biography *John Napier: Life, Logarithms, and Legacy* by the English Winchester College mathematician *Julian Havil*, which serves as a significant source for this essay. See Havil (2014), [2]. Chapters 3 and 4, as well as Appendix L of Haval's book, are particularly relevant to our discussion. The reader will notice that our essay adopts a structure somewhat similar to those chapters. While Havil (2014), [2], translates Napier's largely linguistic mathematics into primarily arithmetic and Euclidean geometric terms, our approach in this essay is more analytically oriented and, we believe, more efficient.

However, it is important to note that we will be using analytical tools from mathematics that did not develop until well after Napier's lifetime, tools of which he was entirely unaware. Nonetheless, given his method of working, it seems likely that he possessed an implicit intuitive understanding of these concepts. Therefore, from a historical standpoint, this essay may not be entirely accurate.

1.4. A revolutionary mathematics book without formulas

Napier's logarithms represented a revolutionary advancement in calculation, as noted by Lynne Gladstone-Millar. In her works from 2003 and 2013, [13], pp. 40 - 42, she eloquently states that both the logarithms themselves and the innovative method Napier employed to derive them were groundbreaking in their impact.

To commemorate the 300th anniversary of the *Descriptio*, a congress convened in Edinburgh in July 1914. During this event, Lord Moulton[4] emphasized the significance of Napier's contributions with great enthusiasm, as Lynne Gladstone-Millar recalls:

> *"No previous work had led up to it, nothing had foreshadowed it or heralded its arrival. It stands isolated, breaking upon human thought abruptly, without borrowing from the works of other intellects or following known lines of mathematical thought".*

See Lynne Gladstone-Millar (2003, 2013), [13], p. 42.

When Napier penned the *Constructio* and *Descriptio* at the dawn of the 17th century, analytic geometry[5] and infinitesimal calculus[6] were not yet developed enough to convey his pioneering mathematical concepts related to *continuously* variable quantities. Additionally, the true nature of the real number system had yet to be unveiled.

When one opens the *Constructio*, one will see (as was not unusual for that time) a mathe-

[3] The first English translation by Edward Wright of the *Descriptio* was already published by the East India Company in 1616, so only two years after the publication of the *Descriptio*, which proves how important logarithms were considered to be for navigational calculations. See [13], p. 44.

[4] Lord Moulton, known as Samuel Moulton, was a prominent British mathematician and civil engineer. He played a significant role in the development of various mathematical theories and applications, particularly in the realms of logarithms and their practical uses in engineering and science.

[5] Descartes' *La Géométrie* did not appear until 1637.

[6] The differential and integral calculus (fluxion calculus, infinitesimal calculus) has its origins in a long article by Leibniz in *Acta Eruditorum* from 1664 and Newton's *Philosophiae Naturalis Principia Mathematica* from 1687. See also [19], pp. 122 ff.; thus, six or seven decades after Napier's work on logarithms.

matics book almost without formulas. Instead, we do see a few drawings with lines that today we would call number lines (-axes), but which for Napier were carriers of line segments, *continuously* changing in length, and not for real numbers as such. Remember, also our familiar coordinate concept was still unknown. His mathematics is therefore mainly geometric in nature.

All of his mathematical descriptions, primarily based on geometry, are written in accessible language — originally Latin (1614), and later translated into English (in particular 1889, see reference [1]). This presents a daunting challenge for today's readers of the *Constructio*, who must expend considerable effort to decipher the text and translate Napier's concepts into contemporary mathematical language. As a result, modern scholars must continually engage with the text, questioning what Napier meant and how his ideas can be articulated using the more familiar, modern mathematical terminology.

In the opening paragraph of the *Constructio*, we find the statement: "It (Napier meant the Logarithmic Table) selected from numbers progressing in *continuous* proportion". Napier's remarkable accomplishment of achieving this revolutionary result, long before the development of continuity mathematics that we will discuss in this essay, is truly commendable.

For more detailed information, please visit Ian Bruce's comprehensive website, *8a, which features a digital version of the *Descriptio* along with translations and annotations. Additionally, you can explore website *8b, which provides a by Ian Bruce reconstructed table of Napier's logarithms.

1.5. Structure of this essay

In Chapter 2 of this essay, we present a concise historical overview of the development of logarithms. John Napier's concept of logarithms, invented in the late 16th century, did not emerge in complete intellectual isolation; it was influenced by earlier thinkers who had an initial, albeit rudimentary, understanding of logarithmic concepts.

It is important to acknowledge the contributions of the Swiss mathematician, instrument maker, and astronomer Jobst Bürgi[7], who developed an *anti-logarithm table* around the same time as John Napier. Bürgi utilized a comparable approach to discretizing positive real numbers, making his work significant in the history of mathematics—independently of Napier's achievements.

Bürgi published his anti-logarithm table in 1620, following persistent encouragement and support from Johannes Kepler. This was six years after the publication of Napier's work. Although Bürgi's logarithm can be considered more comprehensible than Napier's, his table did not gain widespread enthusiasm among users.

One reason for this was probably the lack of sufficient explanations and examples accompanying the tables. See Havil (2014), [2], p. 268. Additionally, the tumult of the Thirty Years' War, which raged in Prague where Bürgi worked for the emperor, likely contributed to the limited adoption of his work. This confusion prevented users from effectively utilizing his tables, leading to a gradual erosion in the European mathematical minds of Bürgi's important contributions to the development of logarithmic theory.

The development of logarithmic calculation after Napier (and Bürgi) was spearheaded by

[7] Jost Bürgi (1552–1632) was a prominent German-Swiss astronomer, mathematician, and instrument maker. He is renowned for his expertise in crafting scientific instruments, his development of mathematical tables—particularly logarithmic and trigonometric tables—and his contributions to astronomy during the late Renaissance period.

Henry Briggs, followed by the Dutch calculators Ezechiël de Decker and Adriaen Vlacq, (see also the next chapter) who finished Briggs' uncomplete logarithm table.

To clarify the relationship between Napier's logarithm *L* and the natural logarithm, we will briefly explore the developments in logarithmic mathematics following Napier's contributions. While Napier made significant strides toward modern mathematical analysis, it is important to note that his logarithm *L* is distinct from the natural logarithm, contrary to common belief. In the remainder of Chapter 2, we will take a moment to address this misconception.

The *Constructio* consists of 59 paragraphs and was not originally intended as a pedagogically sound textbook. Napier's true purpose becomes only evident primarily through the continuous, kinematic model he outlines only in §26 of the *Constructio*, where he clarifies his revolutionary conception of a logarithm.

In Chapter 3 of this essay, we begin with an updated description of the kinematic model presented in that §26, reformulated in contemporary mathematical language. Utilizing this model, we will (re)construct the logarithm table that Napier likely envisioned. Despite the limited calculation techniques of his time, we will observe that Napier achieved remarkable accuracy: his table features only a few occasional errors and inaccuracies, which only become apparent when compared to our modern electronic computations.

Similar to Bürgi's logarithm, Napier's approximation of the exact logarithm *L* was derived through the discretization of the continuum of the positive real numbers. Napier's approach involved a series of numerical approximations of logarithms that, while relatively straightforward by today's standards, he developed without the benefit of modern mathematical tools.

In Chapter 3 of this study, we present a detailed description of the kinematic model, and in Chapter 4, we focus on the discrete approximation model utilized by Napier, which he likely developed independently of Bürgi.

Calculating the logarithms of the Sine function required the development of comprehensive reference tables derived from a variety of geometric and arithmetic sequences. In Chapter 5, we reconstruct a 21 x 69 matrix containing 1,449 reference values, which formed the basis for the subsequent creation of Napier's logarithm table. Napier gathered this remarkable collection of reference values to simplify the process of calculating numerical approximations for the logarithms of the quarter circle, as depicted in Figure 1.2.1.

Finally, in Chapter 6, we will explore how Napier utilized this reference matrix to compile the definitive logarithmic table for trigonometric functions.

Conclusions and references the reader will find in Chapter 7.

CHAPTER 2
A HISTORICAL SKETCH

2.1. Various logarithm tables

In the late 16th and early 17th centuries, calculating logarithms was a remarkable achievement. John Napier became the first to publish a comprehensive logarithm table in his work *Descriptio*, released in 1614. This table was particularly useful for trigonometric calculations, which were essential in fields such as astronomy, navigation, and surveying.

Subsequently, Henry Briggs sought to create a more broadly applicable logarithm table based on Napier's concepts. However, after more than a year of painstaking calculations in 1617, he abandoned his efforts due to exhaustion. Nevertheless, seven years later, in 1624, Briggs published *Arithmetica Logarithmica*, which included decimal logarithms (so, with base 10) for numbers ranging from 1 to 20,000 and from 90,001 to 100,000, providing 14 decimal places of precision. Unfortunately, this table was also incomplete.

The Dutch mathematicians Ezechiël de Decker[8] and Adriaen Vlacq[9] completed the unfinished work of Henry Briggs by finalizing his logarithm table after nearly a decade of tedious calculations, finishing in 1626/1627[10]. See Van Poelje (2005), [9] and Van der Zijden (2000), [10]. Their table became an international sensation, achieving best-seller status! With only minor corrections to arithmetic errors, it served as the foundational reference for numerous logarithm tables for over 350 years, until the advent of electronic calculation tools around 1975 rendered logarithm tables largely obsolete within just a few years. Remarkably, even after all this time, their comprehensive version continues to be referred to as *Briggs' Logarithm Table*.

2.2. Misunderstanding the numerical base of Napier's logarithm

In some texts and tables, the natural logarithm, denoted as ln (*logarithmus naturalis*), is referred to as the *Neperian logarithm*. This terminology can be found in various resources, including military reference books like [3].

The term *Neper* is also occasionally used in the field of electrical engineering. For instance, electronics engineers — particularly those working with transmission lines — utilize the Neper as an alternative to the decibel. In this electricity context, both Briggs' base 10 logarithm and the natural logarithm are applicable for comparing two field quantities: $D = 20 \cdot \log \frac{X_1}{X_2}$ dB and the natural logarithm $N = \ln \frac{X_1}{X_2}$ Np apply. An example of this is the attenuation of a signal in a telephone cable, which can be expressed in for instance Neper per kilometre length (Np/km)[11]. For more information, see King et al. (1945), [4].

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[8] Ezechël de Decker (1603/1604-1646/47), Dutch surveyor and teacher of geometry and arithmetic.

[9] Adriaan Vlacq (1600-1667), Dutch astronomer, mathematician and publisher.

[10] In 1628 Vlacq published his *Arithmetica Logarithmica*, without mentioning his co-worker De Decker.

[11] Compared to the dB, the Neper is hardly used anymore. This is remarkable because so many phenomena in electrical engineering are described with *e*-powers. For example, the potential drop along a transmission cable as

These kinds of texts highlight an intriguing connection between the natural logarithm and John Napier, a connection of historical significance. Napier was a pivotal contributor to the development of logarithms, even coining the term *logarithm* itself (derived from the Greek *logos* meaning ratio and *arithmos* meaning number). He also compiled the second, and notably the first published, logarithm table in his work *Descriptio* in 1614. However, it is important to note that he was *not* the inventor of the natural logarithm, the logarithm with base *e*, which was discovered long after Napier's death in 1617. A notable example of this misunderstanding can be found in Gladstone-Millar (2013), p. 50.

We must clarify: the base of Napier's logarithm is not Euler's number *e*! Despite this, the term *Neperian logarithm* implies a closer relationship between Napier's logarithm and the natural logarithm than actually exists. In Chapter 3, we will explore the true mathematical nature of this connection.

2.3. The hyperbolic logarithm

Only in the 18th century (so, long after Napier) it was discovered that the area under the hyperbola $y = 1/x$, namely $A(x) = \int_1^x \frac{1}{u} du$, has the basal logarithmic property[12] $A(a \cdot b) = A(a) + A(b)$. That is the property that the logarithms of Briggs, found much earlier, have, and before that of Stiffel's[13], but which is certainly not entirely a property of the Napier logarithm *L*, nor of the logarithm of Bürgi[14].

In Leonard Euler's[15] *Introductio in analysin infinitorum* published in 1748, the concepts of powers, numerical bases, and logarithms converge into a unified synthesis. Since then, it has become evident that every logarithm can be expressed in terms of any other, leading to the conclusion that every power, regardless of its arbitrary base, can also be articulated in relation to all others. The natural logarithm has come to be recognized as the fundamental logarithm, serving as a reference function from which logarithms of arbitrary base *ρ* (positive ratio, unequal 1) can be derived using the straightforward Formula:

$$\log_\rho(x) = \frac{\ln x}{\ln \rho};\ \rho > 0;\ \rho \neq 1;\ x > 0 \qquad (2.3.1)^{16}$$

In this essay, we will primarily focus on real numbers, particularly those that are positive. However, thanks to Euler's pioneering work, we could easily extend the concept of logarithms to include negative or complex numbers *x* if that would be relevant.

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a function of its length L is equal to $V_L = V_{in} \exp(-\alpha L) \rightarrow \alpha L = \ln \frac{V_{in}}{V_L}$. By expressing the attenuation coefficient α in Np/km, one avoids the extra calculation step that one has to do with dB's.

[12] The area under the hyperbola was therefore initially called a *hyperbolic* logarithm.

[13] Michael Stiffel (1487 – 1567), German monk and mathematician.

[14] Jost Bürgi (1552 – 1632), Swiss clockmaker and constructor of astronomic instruments

[15] Leonhard Euler (1707-1783), the most important mathematician of the 18th century.

[16] "This formula also shows that it doesn't matter which logarithm one uses as a reference. Analysis textbooks typically use the natural logarithm because it can elucidate the relationship between natural logarithms and hyperbolic and trigonometric functions. In contrast, schoolbooks often prefer to use Briggs' logarithm, which has a base of 10, as a starting point since it aligns better with the decimal system.

2.4. Euler's explosive base number *e*

As might be well-known, the natural logarithm also has a base, namely *e* = 2.71828..., Euler's number, the transcendent numerical base of all exponential growth, with $\ln e = 1$. It is this base number that unites the five basic constants of mathematics in Euler's famous equation $e^{i\pi} + 1 = 0$, an equation that many mathematicians consider the most beautiful analytical expression.

We can call the number e the *explosive base*: the young Euler introduced it in a manuscript on cannons, the *Meditatio in experimenta explosione tormentorum nuper instituta*, written in 1727/1728, more than some hundred years after the death of John Napier, and some hundred years after the Dutch De Decker/Vlacq-publication of the first full decimal logarithm (base 10) table.

We will see that the natural base *e* has something to do with the Napier logarithm *L*. Without realizing it, and thus without mentioning it, Napier had in fact unintentionally discovered a relation between logarithms and the natural base number *e*. But we must emphasize again: the number *e* is *not* the base of Napier's logarithm!

2.5. Prosthaphairesis[17]

Before Napier published his logarithm table, the foundational concept of 'logarithms' had existed for quite some time. According to Havil (2014), [2], p. 62 ff., trigonometric formulas dating back to the Middle Ages were already utilized in astronomy and navigation, enabling the conversion of sums and differences into products and quotients, and vice versa:

$$\begin{cases} \cos(x+y) = \cos x \cos y - \sin x \sin y \\ \cos x \cos y = \dfrac{1}{2}\{\cos(x+y) + \cos(x-y)\} \end{cases} \tag{2.5.1}$$

These formulas are computationally quite intensive.

2.5.1. Arithmetic and geometric sequencies

Around 1500, the mathematicians Michael Stiffel (1487–1567), and Nicolas Chuquet[18] developed a method to convert calculations involving multiplication and division into sums and differences through the use of an obvious connection between arithmetic and geometric sequences. For example, some predecessors of Napier used the powers 2^n, with which they made a connection between the arithmetic sequence of exponents (magnitudes) *n* = ..., -2, -1, 0, 1, 2, ... and the geometric sequence of the powers ..., ¼ , ½. 1, 2, 4, 8, ... See for instance Donners (2002), [5], pp. 59 - 61. Likewise, the history of this early logarithmical concept has extensively been described and discussed in detail by Stephan Weiss. See [6].

In the 16th century, the relationship[19] between the two sequences became more general: the

[17] *Prosthaphaeresis* or *prosthapheiresis* is a neologism coined in the 16th century from the two Greek words prosthesis = addition and aphaeresis = subtraction.

[18] Nicolas Chuquet (1445-1488), French mathematician who introduced many algebraic notations.

[19] The idea of a logarithmic relationship between geometric and arithmetic sequences was not only acknowledged by Michael Stifel but was also recognized by Simon Jacob (circa 1510–1564) and Mauritius Zons (Mauritius van Nassau, 16th–17th century). See [20], p. 43. Stifel's landmark work, *Arithmetica Integra* (1544), introduced significant mathematical notation that marked a pivotal moment in mathematics. Notably, it features the earliest

geometric sequence $\{x_n\} = a, a\rho, a\rho^2, a\rho^3, \ldots$, with various initial factors *a*, and various common *ratios* ρ, was related to an arithmetic sequence $\{\lambda_n\} = b, b+\delta, b+2\delta, \ldots$ of *magnitudes*, or *logarithms* as Napier called them, with various initial terms *b* and various common *differences* δ. This connection was called *prosthaphairesis*. That remarkable word comes from the Greek words *prosthesis* (πρόσθεσις) and *aphaeresis* (ἀφαίρεσις), what indicates at the combination of addition and subtraction in one process.

The modern basic idea of logarithms, that originates in Euler's work, is of course that every positive real number *x* > 0 can be expressed as a power with a constant positive base ρ, unequal 1, and with initial factor *a* = 1, thus as $x = \rho^n$. Possibly, successive integer values of the exponent *n* of ρ (i.e. the logarithms, or the magnitudes of *x*) then form an arithmetic sequence.

The underlying concept is quite straightforward: one can translate complex calculations involving the original positive real numbers *x* into simpler computations using magnitudes *n* (the logarithms). As is well known, multiplication can be transformed into addition, division becomes subtraction, and taking square roots translates to dividing by 2, 3, and so on.

The Swiss clockmaker Jost Bürgi created the first logarithm table (in fact an anti-logarithm table) inspired by that concept; however, he only published it in 1620, after persistent urging from Kepler[20], which was several years after the release of Napier's *Descriptio* in 1614.

2.5.2. Ioannis Kepler

The astronomer Kepler was a passionate advocate for the use of logarithms. Leveraging the more convenient and accurate calculations made possible by logarithmic calculations — likely influenced by Bürgi or Napier — Kepler made important pioneering discoveries in the early 17th century. As readers of this study may already know, in 1609, Johannes Kepler Formulated his first law of planetary motion, which describes the elliptical orbits of planets around the Sun. He subsequently established his second law, which states that a line segment connecting a planet and the Sun sweeps out equal areas in equal intervals of time. In 1619, he introduced his third law, detailing the relationship between the time a planet takes to orbit the Sun and its average distance from the Sun. It may be presumed he used some logarithm concept in all the cumbersome calculations. For more details, please refer to Lombardi (2007), [7], pp. 38-39.

On another note, although Bürgi's logarithm tables were innovative, they were reportedly difficult for users to interpret, leading to their lack of widespread popularity; in fact, only a small number of copies of these tables have survived to this day. For further reading, consult Donners (2002), [5], p. 61 ff.

2.6. Stigler's law of eponyms[21]

While Napier is credited with coining the term *logarithm*, he was not the true discoverer of logarithms themselves. As mentioned, there were earlier mathematicians who grasped the concept in a rudimentary form, particularly through the relationship between geometric and

recorded use of multiplication by magnitudes. Additionally, Stifel is credited with being the first to employ the term *exponent* for magnitude and he articulated key rules for calculating with powers of a constant base number.

[20] Ioannis Kepler (1571 – 1630).

[21] Stephen M. Stigler (1941-) emeritus professor History of Statistics, University of Chicago.

arithmetic sequences. However, Napier introduced a distinctive kinematic model that utilized a *continuous* geometric sequence to define his logarithm — a concept that his predecessors were not familiar with at the time, making it feel remarkably *modern*.

Stigler's Law of Eponymy states that no scientific discovery is named after its original discoverer, a principle that applies to Stigler himself, as the phenomenon had been observed by others long before his time.

This law is also evident in the context of logarithms; for example, what we now refer to as the natural logarithm is also known as the *Neperian logarithm*, despite the fact that the natural logarithm was only formally recognized long after Napier's death. See Stigler, 1980, [8]. The natural logarithm[22], like Euler's natural base number *e*, which we shall use in Chapter 3 of this essay, therefore does not appear in the *Constructio* at all. See Macdonald (1889), [1].

[22] The French call the natural logarithm *logarithme népérien*, an attractive, but historically incorrect name for the abbreviation *ln* for natural logarithm (*logarithmus naturalis*).

CHAPTER 3
NAPIER'S CONTINUOUS KINEMATIC MODEL

3.1. The exact Napier logarithm *L*

Napier conceived a groundbreaking continuous kinematic model as a thought experiment, which ultimately yields a non-integer geometric common ratio, in this essay denoted as ρ. The specific value of ρ will be calculated in the following paragraph.

After two decades of meticulous calculations, this precise continuous model culminated in a discrete table that provides accurate approximations of the exact logarithms *L* of the Sine, Cosine, and Tangent (capitalized) for angles ranging from 90° to 0° (in this order!), measured in increments of one minute of arc. cf. Table 1.2.1.

One of the striking features of Napier's table construction is the initial factor $a = R = 10^7$ of his basal geometric sequence $\{x_n\}$. He was able to work with Stevin's[23] decimal fractions, but found them cumbersome for practical use. Instead, he opted to multiply all fractions by 10^7, which allowed him to express most of the values of his Sines and approximate logarithms using the much more familiar integer parts of seven digits to the left of the decimal point. (Napier is credited with applying our modern decimal and point notation, which he used to enhance accuracy). However, the limitations of this approach became evident in the years 1615 -1616 when Briggs pointed out that it was possible to define a logarithm without the initial factor *R*. He argued that adopting an initial factor of $a = 1$ was a more sensible approach. Unfortunately, at that time Napier was already too old and ill to revise his extensive calculations.

Briggs further recognized that it would be more practical to set the logarithm of the unit 1 equal to 0. He also noted that using base 10 — despite the decimal system not yet being widely adopted — was more intuitive than Napier's combination of the initial factor $R = 10^7$ and the specific geometric common ratio ρ he had devised. Moreover, Briggs understood that a monotonically increasing logarithm would be far more useful than a monotonically decreasing one. And in general, he thought about numbers from small to large, while Napier understood the numbers $x = \mathrm{Sin}\theta = R\sin\theta$ decreasing from large down to small.

Briggs took advantage of Napier's fundamental ideas and began calculating logarithms starting from those more practical insights. However, he stopped calculating after one year. He published his incomplete logarithm table in 1617, the same year that Napier died[24].

As mentioned in Chapter 2 above, that table was supplemented in 1626/27 by the Dutch calculators Ezechiël de Decker and Adriaan Vlacq. Publications by Otto van Poelje (2005), [9] and Thomas van der Zijden (2000), [10], members of the Dutch Circle for Historical Calculating Instruments (KRING), contain interesting details about these two (human) computers[25].

---

[23] Stevinus, Simon Stevin (1548-1620), a Flemish mathematician, engineer, and military engineer. He is known for his contributions to mathematics, particularly in the fields of decimal fractions and the fixed point. He introduced the concept of decimal notation in Europe, which had a significant impact on mathematics and science.

[24] After a visit to Napier in 1615, Briggs wrote his first work on logarithms, *Logarithmorum Chilias Prima*, which was published in 1617. A further mathematical treatise appeared in 1624 under the title *Arithmetica Logarithmica*. This work contained the logarithms of the natural numbers from 1 to 20,000 and from 90,000 to 100,000 calculated to 14 decimal places. He drew up those tables by using pen and paper to draw the first 27 consecutive square roots of 10 with 16 digits after the decimal point.

[25] In 1626 De Decker published a book entitled *Eerste Deel van de Nieuwe Telkonst*. Also, in 1626 De Decker published a second, easier to use, book with the shorter title *Nieuwe Telkonst*. On October 2, 1627, De Decker's

3.2. Napier's logarithm: two points in motion

Following Napier's thoughts, the here constructed continuous, exact, kinematic model produces Napier logarithms $L(x)$ per minute of arc of values $x = \mathrm{Sin}\theta = R\sin\theta$, for angles $90°\,0' \geq \theta \geq 0°\,0'$ and radius $R = 10^7$. [26]. Herein is x the length of the ever-changing interval OX. See Figure 3.2.1.

What is most important to realise is that Napier envisioned a *continuous* geometric sequence, which was a novelty at the time. Nowadays we consider such a sequence an exponential function. We point that out again: the English translation of the *Constructio* underlines that clearly by Napier's remark: "It is picked out from numbers progressing in *continuous* proportion". See Macdonald (1889), [1], p. 7.

3.2.1. Moving in an exponential manner

For the following, see also Van der Salm (1999) [11] and Havil (2014), [2], pp. 96-130.
Napier came up with the following revolutionary idea. See Figure 3.2.1.

1. There are two 'number' lines (as we call them today), along which the points X and L travel.
2. On the upper line, point X, with numerical value *x,* traverses the finite interval OR, with length $R = 10^7$. X moves from right to left, so from the right most point R to the origin O.
3. On the bottom line, point L, with numerical value L (the Napier logarithm of x) extends half a line from origin O to infinity. Point L moves from left to right.
4. So, X starts on the upper line in the point with value $x = R = 10^7$. That starting point is the Sine of 90°. The *variable* velocity of X is proportional to the value $-x$. X's initial velocity is **−***R* (m/s or similar unit). Mind the minus sign in front of *R*! The proportionality factor is -1. See for explanation Formula (3.2.11) hereafter.
5. Point L, with distance $L = L(x)$ to O (the origin of the lower line) traverses the lower line, from O ($L = 0$ for the Sine of 90°), with a *constant* velocity **+***R* (m/s or comparable unit) to $L = \infty$. Mind the plus sign in front of *R*!
6. When X runs on the upper line to the left, the Sinuses $x = \mathrm{Sin}\theta = R\sin\theta$ (= length of OX) become smaller and thus become the angles θ smaller. The further point X is to the left, the smaller its value x is, and the larger the corresponding exact, continuous Napier logarithm $L = L(x)$.

What does this mean in contemporary mathematical terminology, concepts that Napier was not familiar with?

From those six remarks above we conclude for the velocity of point X on the upper line, moving to the left (negative velocity) in the direction of the origin O of the upper line:

Tweede Deel van de Nieuwe Telkonst was published. This book contains a full table with all logarithms of the numbers 1 to 100 000. Only one copy of the *Tweede Deel* is known. It was only discovered in 1920 in the library of a life insurance company in Utrecht, the Netherlands, EU.

The logarithm table in Vlacq's *Arithmetica Logarithmica* is clearly a copy of the Table in the *Tweede Deel*, but Vlacq avoids mentioning De Decker's name in the work. In contrast to the *Tweede Deel*, Vlacq's *Arithmetica Logarithmica* became a great commercial success. See website *4.

[26] 90 degrees correspond to 5,400 minutes of arc; Napier set himself the task of calculating 5,401 logarithms.

$$v_x = \frac{dx}{dt} = -x \tag{3.2.1}$$

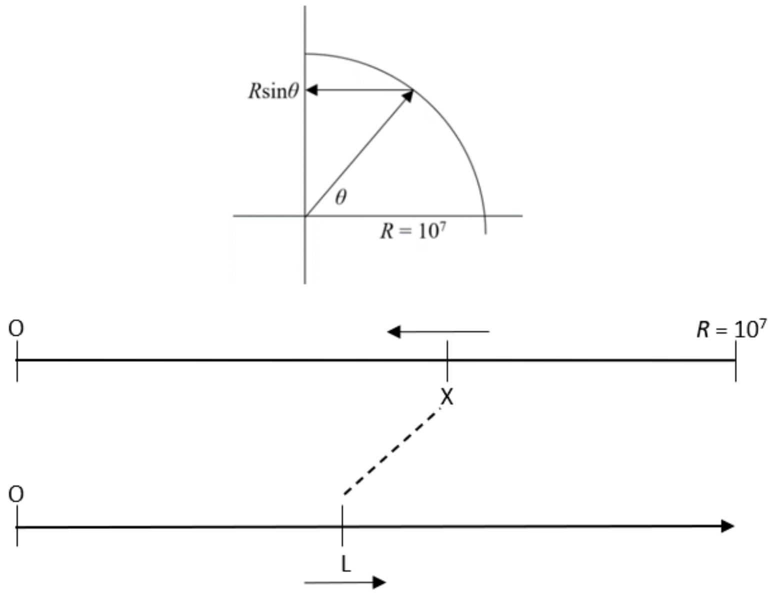


Fig. 3.2.1. Relationship between $x = \text{Sin}\,\theta = R\sin\theta$ ***and the exact Napier logarithm L of x.***

At starting moment *t* = 0, Napier chose velocity $v_X(0) = -R$ (m/s or similar unit). From Formula (3.2.1) (in our contemporary mathematical terms a simple first order differential equation) therefore follows:

$$x = x(t) = R\left(\frac{1}{e}\right)^t \tag{3.2.2}$$

So, the distance *x* (in Napier's thinking the length of the interval OX) is expressed as an exponential function of time with base $\frac{1}{e}$, multiplied by *R* = 10^7. Formula (3.2.2) shows *in modern mathematics* what Napier probably must have meant by a *continuous* geometric sequence.

The relationship in Formula (3.2.2) between time segment *t* and *x*, is:

$$t = \ln\frac{R}{x} \tag{3.2.3}$$

3.2.2. A moving logarithm

At the same moment as X starts moving to the left in point R on the upper line, point L on the bottom line moves with a *constant* velocity +*R* (m/s or similar unit) to the right. Hence, at moment *t*, point L has travelled on the lower line the distance:

$$L = R\,t \leftrightarrow t = \frac{L}{R} \tag{3.2.4}$$

The two formulas of the Napier logarithm L are therefore[27]:

$$x(L) = R\left(\frac{1}{e}\right)^{L/R} \tag{3.2.5}$$

And inversely:

$$\text{Naplog}(x) = L(x) = R \cdot \ln\frac{R}{x} = R \cdot \log_{1/e}\left(\frac{x}{R}\right) \tag{3.2.6}$$[28]

From Formula (3.2.6) we conclude that the exact, continuous Napier logarithm L has admittedly something to do with the natural logarithm, but that those two logarithms are certainly not identical.

3.2.3. A negative velocity?

The answer to the following question is crucial for understanding how Napier calculated logarithms. How might Napier have envisioned the concept that the velocity of point X on the upper line in Figure 3.2.1 is inversely proportional to the distance x from point O to point X? See Formula (3.2.1). Refer to Figure 3.2.1 for clarification.

From the *Constructio* we deduce the following. Napier discretely divides the continuous interval OR into subintervals $(x_{n+1}, x_n) = (R^{n+1}\rho, R^n\rho)$ according to the *decreasing* geometric sequence $\{x_n\} = \{R, R\rho, R\rho^2, R\rho^3, ...\}$, with some common ratio $0 < \rho < 1$, and ρ *very near 1*, so that the distance between two consecutive terms becomes extremely small (almost continuously in Napier's thoughts). See Figure 3.2.2.

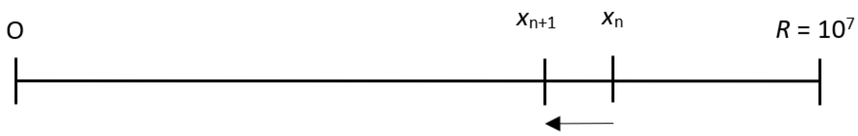


Fig. 3.2.2. The discrete division of the interval OX.

Hence, the already relatively small distance $\Delta(x_n) = x_{n+1} - x_n = R\rho^{n+1} - R\rho^n = R\rho^n(\rho - 1)$, between two consecutive terms of the geometric sequence $\{x_n\} = \{R, R\rho, R\rho^2, R\rho^3, ...\}$, even decreases to smaller values as X gets closer to the origin O.

The time segment Δt, for point X to go left through a subinterval $(R^{n+1}\rho, R^n\rho)$, Napier presumed to have the same length for each subinterval. Hence, the *average* velocity of point X, moving left from x-value $x_n = R\rho^n$ to the next x-value $x_{n+1} = R\rho^{n+1}$ left of it, is therefore proportional to the *negative* x_n because:

[27] The reader should recognize that these 'modern' formulas implicitly articulate the continuity model utilized by Napier, even though he was unaware of them. It is noteworthy that his discretization of this model drew inspiration from the underlying concepts of these very formulas.
[28] The notation *NapLog* is used by Havil and others to distinguish it from the usual log notation. See for instance Chapter 3 of [2].

$$v_n = \left(\frac{dx}{dt}\right)_{x=x_n} \approx v_{avr,n} = \frac{\Delta(x_n)}{\Delta t} = \frac{R\rho^{n+1} - R\rho^n}{\Delta t} = -\left(\frac{1-\rho}{\Delta t}\right)\left(R\rho^n\right) \propto -x_n \tag{3.2.7}$$

See Macdonald (1889), [1], p. 18.

Formula (3.2.7) can be generalised to the continuous differential equation:

$$\frac{dx}{dt} = -\alpha x \tag{3.2.8}$$

Herein the (positive) proportionality factor is:

$$\alpha = \frac{1-\rho}{\Delta t} \tag{3.2.9}$$

From Formula (3.2.7), and the boundary condition $x(0) = R$, follow:

$$x = R\,e^{-\alpha t} \tag{3.2.10}$$

As velocity of point X on the upper line, we find:

$$\frac{dx}{dt} = -\alpha R\,e^{-\alpha t} \tag{3.2.11}$$

For which the following boundary condition yields:

$$\left(\frac{dx}{dt}\right)_{t=0} = -R \tag{3.2.12}$$

Conclusion $\alpha = 1$.

So, if one chooses $R(\rho - 1) = -1$ (as Napier presumably might have done in modern mathematical terms), we arrive at:

$$\Delta(x_n) = x_{n+1} - x_n = R\rho^{n+1} - R\rho^n = R\rho^n(\rho - 1) = -\rho^n = -\left(1 - \frac{1}{R}\right)^n \tag{3.2.13}$$

3.2.4. Napier's basal common ratio

Hence, here the common ratio (very near 1) $\rho = 1 - \frac{1}{R}$, that Napier applied for his basal geometric sequence, appears.

For the terms of the geometric sequence $\{x_n\} = \{R, R\rho, R\rho^2, R\rho^3, \ldots\}$ that discretises the continuous interval OR, we finally find the basal geometric sequence $\{x_n\}$ that was actually applied by Napier:

$$x_n = R\rho^n = R\left(1-\frac{1}{R}\right)^n = 10^7\left(1-\frac{1}{10^7}\right)^n ;n \geq 0 \qquad (3.2.14)$$

We will extensively apply this fundamental Formula (3.2.14), and sometimes its inverse, in Chapter 4 of our study:

$$n = \log_\rho\left(\frac{x_n}{R}\right) \qquad (3.2.15)$$

For further reference, see Havil (2014), [2], pp. 100 ff.

3.2.5. Some calculation rules

Some fundamental calculation formulas of the exact, continuous Napier logarithm $L = L(x)$ that follow from Formula (3.2.6) immediately, are:

$$\begin{cases} L(x_1x_2) = L(x_1) + L(x_2) - L(1) \\ L(\frac{x_1}{x_2}) = L(x_1) - L(x_2) + L(1) \\ L(x^n) = nL(x) - (n-1)L(1) \end{cases} \qquad (3.2.16)$$

That Napier thought par excellence in terms of ratios of sines (or Sines) in spherical trigoniometry is evident from the proportionality of two ratios that he often used (a, b, c and d are x-values in the interval OR on the upper line of the model in Figure 3.2.1):

$$\frac{a}{b} = \frac{c}{d} \leftrightarrow L(a) - L(b) = L(c) - L(d) \qquad (3.2.17)$$

Formula (3.2.17) might presumably be the reason he gave his quantity L the name *logarithm*, which means *ratio number*.

In the domain of spherical triangle calculations, for which Napier initially devised his logarithm table, there are often two equal ratios of two sines (or Sines). Formula (3.2.16) shows that the awkward terms with the logarithm of 1, being $L(1)$, (unequal 0!), right of the equal sign in Formula (3.2.15), cancel each other out. Moreover, the factor R doesn't play any disturbing role anymore, so, Formula (3.2.6) makes a useful calculation possible.

3.2.6. Napier's log is not the natural logarithm *ln*

Once again, we emphasize, Formula (3.2.6), being

$\text{NapLog}(x) = L(x) = -R \cdot \ln\frac{R}{x} = R \cdot \log_{1/e}\left(\frac{x}{R}\right)$, shows that Napier's logarithm L is not identical with the natural logarithm. If we would like to talk about a base of Napier's logarithm, it would have to be - with some hesitation - $1/e$. Not quite, because then one must overlook the initial factor $R = 10^7$, that is unequal to 1, and its logarithm that is unequal to 0.

3.3. Some logarithmic graphs

Figure 3.3.1 shows the values of Napier's logarithm as a function of the distance *x*, i.e. the length of the interval OX:

$$\text{NapLog}(x) = L = L(x) = -R \cdot \ln\frac{R}{x} = R \cdot \log_{1/e}\left(\frac{x}{R}\right) \qquad \text{(3.2.6) and (3.3.1)}$$

Herein is $L(1) \approx 1{,}6 \cdot 10^8$. That value is not only unequal to 0, but is also a relatively large number. See Figure 3.3.1. This non-zero value gets in the way of useful arithmetic properties. See the Formula's (3.2.16).

For logarithms in general we want for instance the basic property $L(x_1 x_2) = L(x_1) + L(x_2)$ instead of $L(x_1 x_2) = L(x_1) + L(x_2) - L(1)$, something that Napier only realized after Briggs made remarks over that.

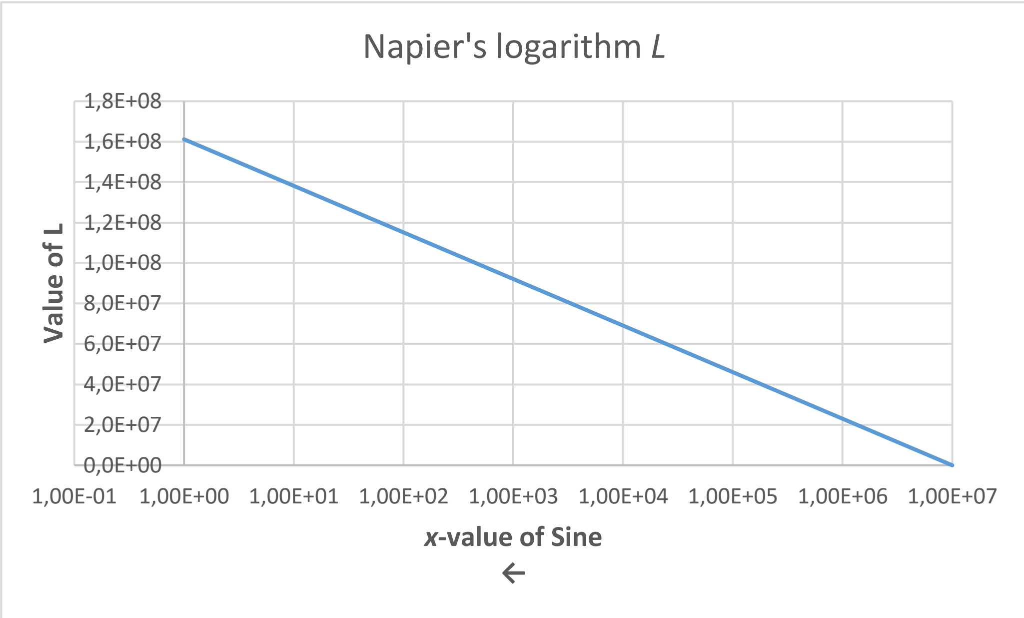


Fig. 3.3.1. Exact, continuous Napier logarithm L(x) against a logarithmic scale. The numbers x on the horizontal axis run from largest to smallest (right to left) according to Napier's left direction of point X in the interval OR in Figure 3.2.1. Note that $L(1) = 10^7 \ln 10^7 \approx 1{,}6 \cdot 10^8$ ***does not equal 0, which makes calculating with Napier's logarithm unnecessarily difficult.***

Figure 3.3.2 plots $x = \text{Sin}\,\theta = R\sin\theta$ on a logarithmic scale against the number of degrees of the angle in the first quadrant. It is striking that from about 6° to 90° all Sinuses *x* lie between 10^6 and 10^7, which corresponds to the rightmost decade in Figure 3.3.1. See also Figure 3.3.3.

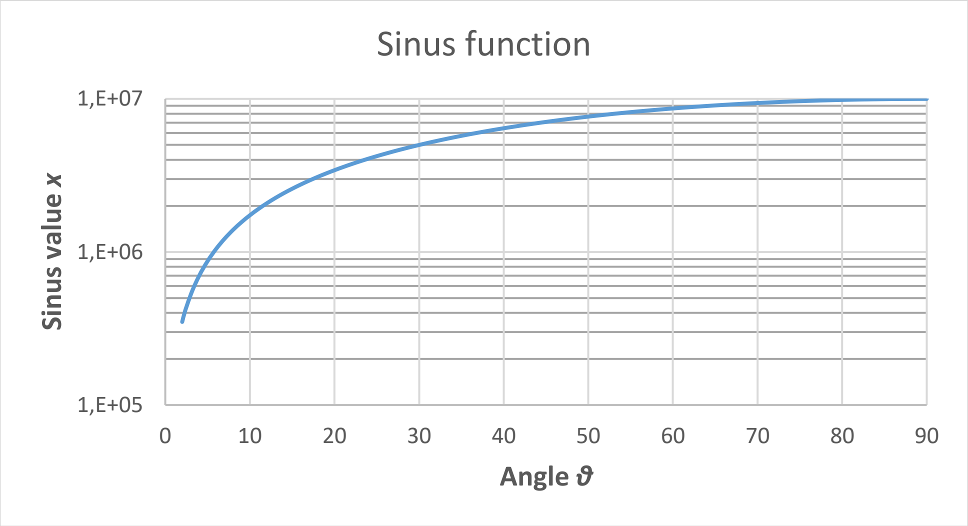
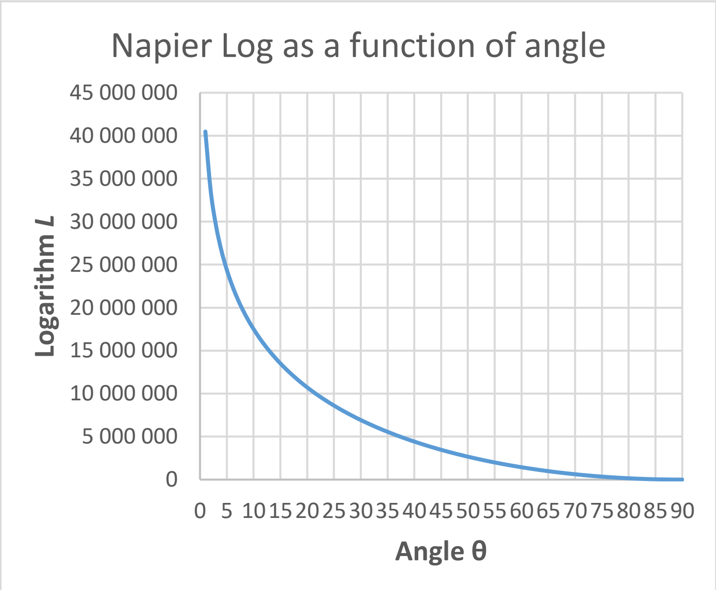


Fig. 3.3.2. The values of the Sine plotted logarithmically against the number of degrees in the first quadrant.

Fig. 3.3.3. Right. The decreasing Napier logarithm as a function of the angle.



Degrees	Sinus	NapLog
0	0.00E+00	
1	1.75E+05	4.05E+07
2	3.49E+05	3.36E+07
3	5.23E+05	2.95E+07
4	6.98E+05	2.66E+07
5	8.72E+05	2.44E+07
6	1.05E+06	2.26E+07
7	1.22E+06	2.10E+07
8	1.39E+06	1.97E+07
9	1.56E+06	1.86E+07
10	1.74E+06	1.75E+07
20	3.42E+06	1.07E+07
30	5.00E+06	6.93E+06
40	6.43E+06	4.42E+06
50	7.66E+06	2.67E+06
60	8.66E+06	1.44E+06
70	9.40E+06	6.22E+05
80	9.85E+06	1.53E+05
81	9.88E+06	1.24E+05
82	9.90E+06	9.78E+04
83	9.93E+06	7.48E+04
84	9.95E+06	5.49E+04
85	9.96E+06	3.81E+04
86	9.98E+06	2.44E+04
87	9.99E+06	1.37E+04
88	9.99E+06	6.09E+03
89	1.00E+07	1.52E+03
90	1.00E+07	0.00E+00

Table 3.3.1. Left. Some values of the Sine and the corresponding Napier logarithms. Most of the 5400 Sinuses have a value between 10^6 and 10^7.

3.4. Examples of table pages with approximations of Napier's logarithms

Gaining a true understanding of the magnitude of Napier's logarithms can be quite challenging. The graphs presented in Paragraph 3.3, along with Table 3.3.1, provide an initial insight into this topic. In Paragraph 3.4, we will explore several worked numerical examples to further illustrate these concepts. Additionally, the Excel logarithm tables below, formatted in the style used by Napier himself, will offer a sense of the scale of the values being calculated.

3.4.1. Example 1

We begin with the Table for an angle θ of 30°, because the Sine of it (= $5 \cdot 10^6$) is situated halfway the interval OR, that is, if we think the interval linearly divided. But if we think the interval OR logarithmically divided, as in Figure 3.3.1, that is a value in de right most decade from 10^6 to 10^7.

Degr	Min	Sin	NapLog	plus/min	NapLog	Sin			Min	Sin	NapLog	plus/min	NapLog	Sin	
30	0	5000000	6931472	5493061	1438410	8660254	60		30	5075384	6781830	5292527	1489303	8616292	30
	1	5002519	6926435	5486345	1440090	8658799	59		31	5077890	6776893	5285876	1491017	8614815	29
	2	5005037	6921402	5479630	1441772	8657344	58		32	5080396	6771960	5279227	1492732	8613337	28
	3	5007556	6916372	5472918	1443454	8655887	57		33	5082901	6767030	5272581	1494449	8611859	27
	4	5010073	6911346	5466208	1445137	8654430	56		34	5085406	6762103	5265937	1496166	8610380	26
	5	5012591	6906322	5459501	1446822	8652973	55		35	5087910	6757179	5259295	1497885	8608901	25
	6	5015107	6901303	5452795	1448507	8651514	54		36	5090414	6752259	5252655	1499604	8607420	24
	7	5017624	6896286	5446092	1450194	8650055	53		37	5092918	6747342	5246017	1501325	8605939	23
	8	5020140	6891273	5439391	1451882	8648595	52		38	5095421	6742428	5239381	1503047	8604457	22
	9	5022655	6886263	5432692	1453571	8647134	51		39	5097924	6737518	5232747	1504770	8602975	21
	10	5025170	6881257	5425996	1455261	8645673	50		40	5100426	6732610	5226116	1506495	8601491	20
	11	5027685	6876254	5419302	1456953	8644211	49		41	5102928	6727706	5219486	1508220	8600007	19
	12	5030199	6871255	5412610	1458645	8642748	48		42	5105429	6722806	5212859	1509947	8598523	18
	13	5032713	6866258	5405920	1460339	8641284	47		43	5107930	6717908	5206234	1511675	8597037	17
	14	5035227	6861265	5399232	1462033	8639820	46		44	5110431	6713014	5199611	1513403	8595551	16
	15	5037740	6856276	5392547	1463729	8638355	45		45	5112931	6708123	5192990	1515133	8594064	15
	16	5040252	6851289	5385863	1465426	8636889	44		46	5115431	6703235	5186371	1516865	8592576	14
	17	5042765	6846306	5379182	1467124	8635423	43		47	5117930	6698351	5179754	1518597	8591088	13
	18	5045276	6841327	5372503	1468823	8633956	42		48	5120429	6693469	5173139	1520330	8589599	12
	19	5047788	6836351	5365827	1470524	8632488	41		49	5122927	6688591	5166526	1522065	8588109	11
	20	5050298	6831378	5359152	1472225	8631019	40		50	5125425	6683716	5159916	1523801	8586619	10
	21	5052809	6826408	5352480	1473928	8629549	39		51	5127923	6678845	5153307	1525538	8585127	9
	22	5055319	6821442	5345810	1475632	8628079	38		52	5130420	6673976	5146701	1527276	8583635	8
	23	5057828	6816479	5339142	1477337	8626608	37		53	5132916	6669111	5140096	1529015	8582143	7
	24	5060338	6811519	5332476	1479043	8625137	36		54	5135413	6664249	5133494	1530755	8580649	6
	25	5062846	6806562	5325812	1480750	8623664	35		55	5137908	6659390	5126894	1532497	8579155	5
	26	5065355	6801609	5319151	1482458	8622191	34		56	5140404	6654535	5120295	1534240	8577660	4
	27	5067863	6796659	5312492	1484168	8620717	33		57	5142899	6649682	5113699	1535983	8576164	3
	28	5070370	6791713	5305835	1485878	8619243	32		58	5145393	6644833	5107105	1537728	8574668	2
	29	5072877	6786770	5299180	1487590	8617768	31		59	5147887	6639987	5100513	1539474	8573171	1
	30	5075384	6781830	5292527	1489303	8616292	30		60	5150381	6635144	5093923	1541222	8571673	0
													Degr	**59**	**Min**

Table 3.4.1. Example page from the 7-digit table with approximations of the exact, continuous Napier logarithms L for the angles from 30° to 31°, divided into 60 minutes, or complementary, the angles from 59° to 60°. Reconstructed with Formula (3.2.6) in Excel.

3.4.2. Remarks about Table 3.4.1

1. The second column indicates a progression of the angle è, starting from 30° and increasing to 31°, incrementing by 1 minute of arc ($1/60^{th}$ of a degree).
2. The eighth column presents (in red) the complementary angle to θ, denoted as (90° - θ), indicating that for each corresponding value of θ in the second column, there is a value in this column that starts from 59° and goes up to 60°, again divided into 60 minutes of arc.
3. The third column calculates the Sine (remark the capital letter) of the angle θ in the second column according the Formula:

$$x = \mathrm{Sin}(\theta) = 10^7 \cdot \sin\left(\frac{\pi}{180}\left(Degr + \frac{Min}{60}\right)\right) \tag{3.4.1}$$

4. The fourth column states, in seven significant digits, the *exact*, continuous Napier logarithm *L* of that Sine:

$$\begin{cases} \text{Naplog}(x) = L(x) = R \cdot \ln \dfrac{R}{x} = R \cdot \log_{1/e}\left(\dfrac{x}{R}\right) \\ L(x) = 10^7 \cdot \ln\left(\dfrac{\text{Sin}(\theta)}{10^7}\right) \end{cases} \tag{3.4.2}$$

5. The sixth column presents the Cosine of θ from top to bottom, with the Sine of the complementary angle (90° - θ) arranged from bottom to top;

$$x = \text{Cos}(\theta) = 10^7 \cdot \cos\left(\frac{\pi}{180}\left(Degr + \frac{Min}{60}\right)\right) \tag{3.4.3)}$$

6. The fifth column, labelled "plus/minus," represents the difference between the Napier logarithms of the Sine values (found in the fourth column) and the Cosine values (in the sixth column). Consequently, this column reflects the logarithm of the tangent of the angle θ, as indicated, from top to bottom, in the first and second columns.
7. As the angle θ increases from 30° to 31°, we observe a decrease in the Napier logarithm, dropping from approximately 6.93 × 10^6 to 6.64 × 10^6.

3.4.3. Example 2

Degr	Min	Sin	NapLog	plus/min	NapLog	Sin			Min	Sin	NapLog	plus/min	NapLog	Sin	
14	0	2419219	14191404	13889860	301544	9702957	60		30	2503800	13847755	13524048	323707	9681476	30
	1	2422041	14179744	13877474	302270	9702253	59		31	2506616	13836514	13512054	324460	9680748	29
	2	2424863	14168099	13865102	302996	9701548	58		32	2509432	13825286	13500073	325213	9680018	28
	3	2427685	14156468	13852744	303724	9700842	57		33	2512248	13814072	13488104	325968	9679288	27
	4	2430507	14144851	13840399	304452	9700136	56		34	2515063	13802871	13476148	326723	9678557	26
	5	2433329	14133249	13828068	305181	9699428	55		35	2517879	13791684	13464204	327480	9677825	25
	6	2436150	14121661	13815750	305912	9698720	54		36	2520694	13780510	13452273	328237	9677092	24
	7	2438971	14110088	13803445	306643	9698011	53		37	2523508	13769349	13440354	328995	9676358	23
	8	2441792	14098528	13791153	307375	9697301	52		38	2526323	13758202	13428448	329754	9675624	22
	9	2444613	14086983	13778875	308108	9696591	51		39	2529137	13747068	13416554	330514	9674888	21
	10	2447433	14075452	13766610	308841	9695879	50		40	2531952	13735947	13404672	331275	9674152	20
	11	2450254	14063935	13754359	309576	9695167	49		41	2534766	13724839	13392802	332037	9673415	19
	12	2453074	14052432	13742120	310312	9694453	48		42	2537579	13713744	13380945	332799	9672678	18
	13	2455894	14040943	13729895	311048	9693740	47		43	2540393	13702663	13369100	333563	9671939	17
	14	2458713	14029469	13717683	311786	9693025	46		44	2543206	13691595	13357267	334327	9671200	16
	15	2461533	14018008	13705484	312524	9692309	45		45	2546019	13680539	13345447	335093	9670459	15
	16	2464352	14006561	13693298	313263	9691593	44		46	2548832	13669497	13333638	335859	9669718	14
	17	2467171	13995128	13681125	314003	9690875	43		47	2551645	13658468	13321842	336626	9668977	13
	18	2469990	13983709	13668965	314744	9690157	42		48	2554458	13647452	13310058	337394	9668234	12
	19	2472809	13972304	13656818	315486	9689438	41		49	2557270	13636449	13298285	338163	9667490	11
	20	2475627	13960913	13644684	316229	9688719	40		50	2560082	13625458	13286525	338933	9666746	10
	21	2478445	13949536	13632563	316973	9687998	39		51	2562894	13614481	13274777	339704	9666001	9
	22	2481263	13938172	13620455	317717	9687277	38		52	2565705	13603517	13263041	340476	9665255	8
	23	2484081	13926822	13608359	318463	9686555	37		53	2568517	13592565	13251317	341249	9664508	7
	24	2486899	13915486	13596277	319209	9685832	36		54	2571328	13581626	13239604	342022	9663761	6
	25	2489716	13904163	13584207	319957	9685108	35		55	2574139	13570700	13227904	342796	9663012	5
	26	2492533	13892855	13572150	320705	9684383	34		56	2576950	13559787	13216215	343572	9662263	4
	27	2495350	13881559	13560105	321454	9683658	33		57	2579760	13548887	13204538	344348	9661513	3
	28	2498167	13870278	13548074	322204	9682931	32		58	2582570	13537999	13192874	345125	9660762	2
	29	2500984	13859010	13536055	322955	9682204	31		59	2585381	13527124	13181220	345903	9660011	1
	30	2503800	13847755	13524048	323707	9681476	30		60	2588190	13516261	13169579	346682	9659258	0
													Degr	**75**	**Min**

Table 3.4.2. Example with a relatively small angle. Page from the 8-digit table (8 Figures left of the decimal point) with approximations of the exact, continuous Napier logarithms L for the angles from 14° to 15°, divided into 60 minutes, or complementary, the angles from 75° to 76°. Reconstructed with Formula (3.2.6) in Excel.

Table 3.4.2 follows the same layout as Table 3.4.1. The equations referenced in the previous example are also applicable to this table. The main difference is the inclusion of a relatively small angle in this instance.

As the angle θ increases from 14° to 15°, we observe the Napier logarithm decreasing from 1.42×10^7 to 1.35×10^7. These values exceed the number 10^7 and thus cannot be represented as a 7-digit integer. Therefore, in contrast to Napier's original presentation, we have chosen to display the logarithms in Table 3.4.2 with 8 significant digits.

3.4.4. Example 3

It is fascinating to examine the differences between our Excel table for an angle of 0 degrees and Napier's original table, presented in Table 1.2.1. The original Table 1.2.1, referenced earlier, features angles θ ranging from 0° to 1°, and their complementary angles from 89° to 90°, all calculated by Napier. Notably, the logarithm of 1 arcminute is recorded as 8.14×10^7, while the logarithm of 1 degree is 4.05×10^7. These values cannot be represented as seven-digit integers. This prompts the question: how did Napier approach this calculation?

Table 3.4.3 displays the Excel version of this table, showing some minor discrepancies between the two. However, it is striking how remarkably accurate Napier's original table remains. It is evident that Napier derived his Table 1.2.1 by dividing each calculated value in Table 3.4.3 by 10; applying this method also leads us to the figures in Table 3.4.4.

Degr	Min	Sin	NapLog	plus/min	NapLog	Sin			Min	Sin	NapLog	plus/min	NapLog	Sin	
0	0	0			0	10000000	60		30	87265	47413868	47413488	381	9999619	30
	1	2909	81425715	81425715	0	10000000	59		31	90174	47085979	47085572	407	9999593	29
	2	5818	74494244	74494242	2	9999998	58		32	93083	46768501	46768067	433	9999567	28
	3	8727	70439594	70439590	4	9999996	57		33	95992	46460793	46460332	461	9999539	27
	4	11636	67562774	67562767	7	9999993	56		34	98900	46162273	46161784	489	9999511	26
	5	14544	65331340	65331329	11	9999989	55		35	101809	45872407	45871889	518	9999482	25
	6	17453	63508126	63508110	15	9999985	54		36	104718	45590709	45590160	548	9999452	24
	7	20362	61966621	61966600	21	9999979	53		37	107627	45316729	45316150	579	9999421	23
	8	23271	60631309	60631282	27	9999973	52		38	110535	45050057	45049446	611	9999389	22
	9	26180	59453481	59453447	34	9999966	51		39	113444	44790313	44789670	644	9999357	21
	10	29089	58399878	58399836	42	9999958	50		40	116353	44537146	44536469	677	9999323	20
	11	31998	57446780	57446728	51	9999949	49		41	119261	44290232	44289520	711	9999289	19
	12	34907	56576669	56576608	61	9999939	48		42	122170	44049268	44048522	746	9999254	18
	13	37815	55776246	55776174	72	9999928	47		43	125079	43813975	43813193	782	9999218	17
	14	40724	55035170	55035087	83	9999917	46		44	127987	43584092	43583273	819	9999181	16
	15	43633	54345245	54345150	95	9999905	45		45	130896	43359376	43358519	857	9999143	15
	16	46542	53699864	53699756	108	9999892	44		46	133805	43139600	43138704	895	9999105	14
	17	49451	53093623	53093500	122	9999878	43		47	136713	42924551	42923616	935	9999065	13
	18	52360	52522043	52521906	137	9999863	42		48	139622	42714030	42713055	975	9999025	12
	19	55268	51981376	51981224	153	9999847	41		49	142530	42507851	42506835	1016	9998984	11
	20	58177	51468449	51468280	169	9999831	40		50	145439	42305838	42304780	1058	9998942	10
	21	61086	50980553	50980367	187	9999813	39		51	148348	42107826	42106725	1100	9998900	9
	22	63995	50515359	50515154	205	9999795	38		52	151256	41913659	41912515	1144	9998856	8
	23	66904	50070848	50070624	224	9999776	37		53	154165	41723192	41722004	1188	9998812	7
	24	69813	49645258	49645015	244	9999756	36		54	157073	41536286	41535052	1234	9998766	6
	25	72721	49237045	49236781	264	9999736	35		55	159982	41352810	41351530	1280	9998720	5
	26	75630	48844845	48844559	286	9999714	34		56	162890	41172641	41171314	1327	9998673	4
	27	78539	48467449	48467141	308	9999692	33		57	165799	40995661	40994286	1375	9998625	3
	28	81448	48103781	48103449	332	9999668	32		58	168707	40821760	40820336	1423	9998577	2
	29	84357	47752876	47752520	356	9999644	31		59	171616	40650832	40649359	1473	9998527	1
	30	87265	47413868	47413488	381	9999619	30		60	174524	40482777	40481254	1523	9998477	0
							Min						**Degr**	**89**	**Min**

Table 3.4.3. This would be the first page with approximations of the exact, continuous Napier logarithms in 8 digits of trigonometric quantities, calculated with Excel, comparable to the Table 1.2.1 actually compiled by Napier.

3.4.5. Napier's decimal point[29]

Additionally, Tables 1.2.1 and 3.4.4 showcase an important innovation that had seen limited adoption up to that point: the decimal point. This notation for decimal fractions, still widely used today, was introduced in certain tables of the *Constructio*. Napier adopted this approach because he found Stevin's fractions too cumbersome for calculations.

Degr	Min	Sin	NapLog	plus/min	NapLog	Sin			Min	Sin	NapLog	plus/min	NapLog	Sin	
0	0	0			0.0	1000000.0	60		30	8727	4741387	4741349	38.1	999961.9	30
	1	291	8142572	8142572	0.0	1000000.0	59		31	9017	4708598	4708557	40.7	999959.3	29
	2	582	7449424	7449424	0.2	999999.8	58		32	9308	4676850	4676807	43.3	999956.7	28
	3	873	7043959	7043959	0.4	999999.6	57		33	9599	4646079	4646033	46.1	999953.9	27
	4	1164	6756277	6756277	0.7	999999.3	56		34	9890	4616227	4616178	48.9	999951.1	26
	5	1454	6533134	6533134	1.1	999998.9	55		35	10181	4587241	4587189	51.8	999948.2	25
	6	1745	6350813	6350813	1.5	999998.5	54		36	10472	4559071	4559016	54.8	999945.2	24
	7	2036	6196662	6196662	2.1	999997.9	53		37	10763	4531673	4531615	57.9	999942.1	23
	8	2327	6063131	6063131	2.7	999997.3	52		38	11054	4505006	4504945	61.1	999938.9	22
	9	2618	5945348	5945348	3.4	999996.6	51		39	11344	4479031	4478967	64.4	999935.7	21
	10	2909	5839988	5839988	4.2	999995.8	50		40	11635	4453715	4453647	67.7	999932.3	20
	11	3200	5744678	5744678	5.1	999994.9	49		41	11926	4429023	4428952	71.1	999928.9	19
	12	3491	5657667	5657667	6.1	999993.9	48		42	12217	4404927	4404852	74.6	999925.4	18
	13	3782	5577625	5577625	7.2	999992.8	47		43	12508	4381397	4381319	78.2	999921.8	17
	14	4072	5503517	5503517	8.3	999991.7	46		44	12799	4358409	4358327	81.9	999918.1	16
	15	4363	5434524	5434524	9.5	999990.5	45		45	13090	4335938	4335852	85.7	999914.3	15
	16	4654	5369986	5369986	10.8	999989.2	44		46	13380	4313960	4313870	89.5	999910.5	14
	17	4945	5309362	5309362	12.2	999987.8	43		47	13671	4292455	4292362	93.5	999906.5	13
	18	5236	5252204	5252204	13.7	999986.3	42		48	13962	4271403	4271306	97.5	999902.5	12
	19	5527	5198138	5198138	15.3	999984.7	41		49	14253	4250785	4250684	101.6	999898.4	11
	20	5818	5146845	5146845	16.9	999983.1	40		50	14544	4230584	4230478	105.8	999894.2	10
	21	6109	5098055	5098055	18.7	999981.3	39		51	14835	4210783	4210673	110.0	999890	9
	22	6399	5051536	5051536	20.5	999979.5	38		52	15126	4191366	4191252	114.4	999885.6	8
	23	6690	5007085	5007085	22.4	999977.6	37		53	15416	4172319	4172200	118.8	999881.2	7
	24	6981	4964526	4964526	24.4	999975.6	36		54	15707	4153629	4153505	123.4	999876.6	6
	25	7272	4923705	4923705	26.4	999973.6	35		55	15998	4135281	4135153	128.0	999872	5
	26	7563	4884485	4884485	28.6	999971.4	34		56	16289	4117264	4117131	132.7	999867.3	4
	27	7854	4846745	4846745	30.8	999969.2	33		57	16580	4099566	4099429	137.5	999862.5	3
	28	8145	4810378	4810378	33.2	999966.8	32		58	16871	4082176	4082034	142.3	999857.7	2
	29	8436	4775288	4775288	35.6	999964.4	31		59	17162	4065083	4064936	147.3	999852.7	1
	30	8727	4741387	4741387	38.1	999961.9	30		60	17452	4048278	4048125	152.3	999847.7	0
							Min						**Degr**	**89**	**Min**

Table 3.4.4 presents a reconstruction of the first page of the Table in the Constructio, featuring approximations of the exact, continuous Napier logarithms for trigonometric quantities calculated to seven decimal places using Excel. These values closely align with the seven-digit table (Table 1.2.1) originally compiled by Napier. Please note the placement of the decimal point and the use of decimal notation for the figures to the right of it, a notation not much used before Napier.

3.5. Two useful inequalities

Every entry in a logarithm table represents an approximation of an exact, continuous real value. In constructing his logarithm tables, Napier employed several inequalities that he discovered without the benefit of the differential and integral calculus that we now consider standard. In the following chapters, we will utilize these inequalities for calculations within a discrete model.

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[29] Napier is renowned for popularizing the use of the decimal point. While it had previously appeared in mathematics, it was not widely adopted until his influence. It appears that the history of the decimal point is more extensive than previously believed. Traditionally, the earliest documented use of a period as a decimal point was attributed to the German mathematician Christopher Clavius in 1593, so, shortly before John Napier's work on logarithms. However, recent research suggests that Clavius may have been drawing from a much older concept that had been largely forgotten. This new study indicates that the decimal point actually dates back to the years 1440 - 1450, approximately 150 years earlier than Napier's writing of the *Constructio,* when it first appeared in the writings of Italian mathematician Giovanni Bianchini (1410-1469). For more information, see websites *5 and *6.

3.5.1. The first logarithmic inequality

A well-known inequality from calculus is:

$$Z > 1 \rightarrow \frac{Z-1}{Z} < \ln Z < Z-1 \qquad (3.5.1)$$

The proof of Formula (3.5.1) is relatively straightforward.

For the ratio of the length R of interval OR and the length x of the interval OX, we find by applying Formula (3.5.1):

$$Z = \frac{R}{x} \quad \rightarrow \quad R - x < L(x) < \frac{R}{x}(R-x) \qquad (3.5.2)$$

With Formula (3.5.2) we can estimate the values of a Napier logarithm L on the lower line of Figure 3.1.1 with x-values (Sines) on the upper line.

3.5.2. The second logarithmic inequality

The next inequality involves two points (Sines) on the upper line in Figure 3.1.1. We choose point X_1 (with x-value x_1 = distance to O) right of point X_2 (with x-value x_2 = distance to O). See Figure 3.5.1.

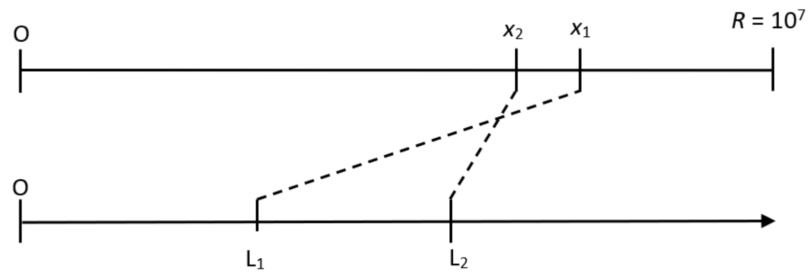

***Figure 3.5.1** This figure illustrates the relationship between two consecutive geometric terms (Sines) and their corresponding logarithms. It's important to note that as the values of x move from right to left, their logarithms L shift from left to right. Additionally, Napier's logarithmic function is monotonically decreasing.*

$$X_1 \text{ right of } X_2 \rightarrow x_1 > x_2 \rightarrow L(x_1) < L(x_2) \qquad (3.5.3)$$

Then $X = \frac{x_1}{x_2} > 1$. If we substitute this into the inequality in Formula (3.5.1), we find:

$$\frac{x_1 - x_2}{x_1} < \ln \frac{x_1}{x_2} < \frac{x_1 - x_2}{x_2} \qquad (3.5.4)$$

Applying Formula (3.2.6), $\text{Naplog}(x) = L(x) = R \cdot \ln \frac{R}{x} = R \cdot \log_{1/e}\left(\frac{x}{R}\right)$, results in:

$$L(x_2) - L(x_1) = R\ln\frac{R}{x_2} - R\ln\frac{R}{x_1} = R\ln\frac{x_1}{x_2} \tag{3.5.5}$$

In combination with Formula (3.5.4) we find the inequality:

$$x_1 > x_2 \;\; \rightarrow \; R\cdot\frac{x_1 - x_2}{x_1} < L(x_2) - L(x_1) < R\cdot\frac{x_1 - x_2}{x_2} \tag{3.5.6}$$

With Formula (3.5.6) we can estimate the difference between two Napier logarithms *L* on the lower line in Figure 3.1.1 with *x*-values on the upper one.

CHAPTER 4
NAPIER'S DISCRETE MODEL

4.1. A discrete division of the interval OR

The modern, precise Formulas (3.2.5) and (3.2.6) interpret the interval OR on the upper line in Figure 3.1.1 as a continuum. However, it's essential for the reader to understand that these formulas represent *our* contemporary mathematical interpretation of the continuum concept, which Napier likely approached only intuitively. At the time, he could not have been aware of these concepts, as the requisite differential and integral calculus (infinitesimal or fluxion calculus) had yet to be developed. Nevertheless, these formulas, along with the two inequalities in Paragraph 3.5, demonstrate how far ahead of his time Napier's thinking was; he was on the edge of modern continuity mathematics. For further details, see for instance Boyer (1959), [19], p. 122.

4.1.1. The essence

What was the essence of Napier's continuity concept? He employed a fine-grained, discrete geometric division $\{x_n\}$, with values $x_n = R\left(1-\frac{1}{R}\right)^n = 10^7\left(1-\frac{1}{10^7}\right)^n$; $n = 0,1,2,\ldots$, as represented by the terms in Formula (3.2.14) corresponding to the interval OR on the upper line of Figure 3.1.1. This method can be viewed as an *almost* continuous approach. However, because this discretization is merely an approximation of the continuum, it necessitates several sophisticated approximations to compute the relevant logarithms of the Sines. For further details, refer also to Havil (2014), [2], pp. 105-107. In this chapter, we will explore these approximations in more depth. So, how 'good' are the logarithms of the terms x_n of the geometric sequence?

4.1.2. Geometric and arithmetic sequencies

We know from calculus $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}$. Due to the large value of the initial factor $R = 10^7$, we can derive the following approximation for the reciprocal of Euler's natural base number, accurate to seven decimal places:

$$\left(1-\frac{1}{R}\right)^R = 0.3678794\ldots \approx \frac{1}{e} = 0.3678794\ldots \qquad (4.1.1)$$

It applies, but Napier (obviously) could not be aware of that.

By translating the Napier logarithm L in Formula (3.2.5), represented on the lower line of Figure 3.1.1, back to the x-values in the interval OR on the upper line, and applying Formula (4.1.1), we can derive the following approximation for the x-values of the continuous logarithms $L = L(x)$:

$$x(L) = R\left(\frac{1}{e}\right)^{\frac{L}{R}} \approx R\left[\left(1-\frac{1}{R}\right)^R\right]^{\frac{L}{R}} = R\left(1-\frac{1}{R}\right)^L \qquad (4.1.2)$$

Napier realized that the basal, decreasing geometric sequence $\{x_n\}$ [30], with terms:

$$x_n = R\left(1-\frac{1}{R}\right)^n ; n = 0,1,2,... \qquad (4.1.3)$$

with common ratio $\rho = \left(1-\frac{1}{R}\right)$, and initial factor R, can serve very well as a discrete, fine-grained classifier of the continuous interval OR. Mind from right to left!

Due to the large value of $R = 10^7$, the common ratio ρ is very close to 1. This allowed Napier to envision the discrete division of the interval OR on the upper axis in Figure 3.2.1 as an almost continuous filling of that interval, particularly within the rightmost decade, which spans from 10^6 to $10^7 = R$.

The natural numbers n, the magnitudes of the x_n, then (ideally) form a discrete, arithmetic sequence $\{L_n\}$ of increasing logarithms L_n on the lower axis. Here we perceive again the then already long existing connection between geometric sequences and arithmetic ones.

From Formula (4.1.3) a straightforward recursive relationship follows:

$$x_n = x_{n-1} - \frac{1}{R}x_{n-1} \qquad (4.1.4)$$

Napier did not have any base in mind for his logarithm, as we modern mathematicians would have, because of our historically later developed knowledge of logarithms; he just found the accuracy of the decimal approximations of Formula (4.1.4), of the values $x = \mathrm{Sin}\theta = R\sin\theta$ in the interval OR, attractive.

Thus, each subsequent term of the basal geometric sequence $\{x_n\}$ in Formula (4.1.3) can be calculated with minimal effort (dividing by $R = 10^7$ = shifting the decimal point over 7 digits to the left, and thereafter calculating a simple difference). See Macdonald (1889), [1], pp. 12 ff.

If we denote the exact, continuous Napier logarithm L, that corresponds to x_n of the geometric sequence on the upper line, by $L(x_n)$ on the lower line, then the 2nd order Maclaurin approximation of that logarithm yields:

$$\begin{aligned} L(x_n) &= R\ln\frac{R}{R\left(1-\frac{1}{R}\right)^n} = -nR\ln\left(1-\frac{1}{R}\right) = -nR\left(-\frac{1}{R}-\frac{1}{2}\left(-\frac{1}{R}\right)^2+...\right) \\ &\approx n\left(1+\frac{1}{2R}\right) = \lambda(x_n) = \lambda_n \end{aligned} \qquad (4.1.5)^{31}$$

[30] We will see that Napier used several geometric sequences. But this basal sequence is always starting point.

[31] Although Napier did not have the 'modern' Formulas of the kinematic model, he discovered that his logarithms of the geometric terms in OR are somewhat unequal to the exponents n = 1, 2, 3, ... of the common ratio ρ of the basal geometric sequence (n = 0 is an exception). See [1], p. 21 and [2], p. 102 and p. 107. That Napier came to that conclusion without the tools of modern analysis is truly astonishing.

Hence, the exact, continuous Napier logarithms $L(x_n)$ on the lower line, of the terms of the basal geometric sequence $\{x_n\}$ on the upper one, form *approximatively* an arithmetic sequence $\{\lambda_n\}$ with common difference $\delta = 1 + \frac{1}{2R} = 1.00000005 = \delta_1$ and with initial term 0[32].

Unfortunately, the non-zero logarithms in Formula (4.1.5) appear to be *only* approximately equal to the non-negative integers *n*, which are the exponents (magnitudes) of the common ratio *ρ* in fact.

Thus, in the end we have identified the following pair of Formulas for Napier's discrete model:

$$\begin{cases} x_n = R\rho^n; x_0 = R = 10^7 \text{ and } \rho = 1 - \frac{1}{R} \\ \lambda_n = n\left(1 + \frac{1}{2R}\right); n = 0,1,2,... \end{cases} \qquad (4.1.6)^{33}$$

4.1.3. Density of the geometric terms

The great density of the geometric terms $x_n = R\left(1 - \frac{1}{R}\right)^n$ in the first decade left of the right-most point R on the upper line, is evident from the calculation:

$$R\left(1 - \frac{1}{R}\right)^n = \frac{1}{10}R \to n = \frac{-1}{\log_{10}\left(1 - \frac{1}{R}\right)} \approx 2{,}3 \cdot 10^7 \qquad (4.1.7)$$

To reach the left border at the value 10^6 of the rightmost decade $(10^6, 10^7)$ of the interval OR, we need to take approximately 23 million geometric steps from right to left! An enormous number. Refer to Figure 3.3.1 for illustration.

Note that for angles ranging from approximately 6° to 90°, the trigonometric values $x = \text{Sin}\theta = R\sin\theta$ all fall between 10^6 and 10^7, as illustrated in Figures 3.3.2 and 3.3.3. This demonstrates the close alignment between the discrete arrangement of the basal geometric sequence, defined by the terms in Formula (4.1.3), and the continuous kinematic model.

But only a total of 90 x 60 = 5,400 logarithms needed to be determined (strictly speaking, 5,401 when including 0).

The precise Napier logarithms *L* listed in our Excel tables range from 0 for an angle of 90° - 0' to approximately 8.14 x 10^7 for an angle of 0° + 1'. How did Napier select the correct 5,401 logarithms from this vast array of available values?

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32 We distinguish between *approximate arithmetic* logarithms $\lambda_n = \lambda(x_n)$ and the *exact continuous* logarithms $L_n = L(x_n)$ of the continuous kinematic model.

33 Here it is clear that Napier, despite his kinematic model, did not think in terms of a base for his logarithm. It was only through comments by Henry Briggs in 1615 that Napier realized that it is more convenient to define a logarithm such that the logarithm of the base 10 is 1 and the logarithm of 1 is 0. See [1], pp. 48 et seq.

We think, Napier must have asked himself the question: if a geometric term $x_n = R\left(1-\frac{1}{R}\right)^n$ is a discrete approximation for an exact value $x = \mathrm{Sin}\,\theta = R\sin\theta$, which integer exponent *n* (specifically, the magnitude or logarithm) then corresponds to that particular *x*-value? Answering this question was not as straightforward as it may seem, especially given that he lacked any mechanical or electronic means of computation. Attempting to sift through the vast array of possible values proved to be a futile endeavour. So, what steps can be taken to advance this inquiry? We will explore Napier's solution in the next chapter of this essay.

4.1.4. Example: the first term of the geometric sequence

As an example, we show how Napier calculated the first logarithm, so, the logarithm with rank number *n* = 1. We remember, Napier states $L(R) = L(10^7) = 0$.

The first geometric term of $\{x_n\}$ in the interval OR, left of the starting point R, is $x_1 = R\left(1-\frac{1}{R}\right)^1 = 999\,999.000\,000\,0$. With inequality Formula (3.5.2), $R - x < L(x) < \frac{R}{x}(R-x)$, we find $1.000\,000\,0 < L(x_1) < 1.000\,000\,1$. Napier equates the midpoint of the two boundaries to be the best arithmetic approximation of the continuous logarithm $L(x_1)$:

$$\lambda_1 = 1.000\,000\,05 = 1 + \frac{1}{2R} = 1 + \frac{1}{2\cdot 10^7} \tag{4.1.8}$$

As expected, this arithmetic logarithm is slightly bigger than the integer 1. See Formula (4.1.6), and see Macdonald (1889), [1], pp. 22-23.

Notice, the continuous logarithm, calculated with the exact $L(x) = R\cdot\ln\frac{R}{x}$ in Formula (3.2.5), is approximately 0.999 999 95, so slightly smaller than 1. The difference between the calculated approximating logarithm and the exact value of the logarithm is only 10^{-7}!

4.2. Bürgi's discrete model: an alternative?[34]

It is noteworthy that during the late 16th century, Jost Bürgi in Kassel, in what is now the central part of Germany[35], more or less around the same time that Napier began constructing his logarithmic tables, developed a more or less similar discrete model for anti-logarithms. Bürgi's model used an initial constant of $a = 10^8$ instead of Napier's $R = 10^7$, and featured the significantly larger common ratio $1+\frac{1}{10^4}$. See for instance Staudacher (2018), [21] pp. 205-

[34] Bürgi's logarithm table, created by the Swiss mathematician Jost Bürgi in the late 16th century, is one of the earliest logarithm tables used for calculations in (applied) mathematics. Bürgi developed his logarithm system independently of John Napier, who is often credited with the invention of logarithms around the same time. Though Napier's logarithms became more widely known due to their publication and dissemination, Bürgi's contributions are recognized as an important part of the history of calculational mathematics. See Staudacher (2018), [21], Chapter 11.

[35] We write the 'modern' name Germany, but Kassel belonged to the Holy Roman Empire around the year 1600. The Holy Roman Empire of the German Nation was a complex political entity that existed from the early Middle Ages until the early 19th century, consisting primarily of German-speaking areas in Central Europe.

215 or Schlüssel zum Kosmos (2023), [22], p. 99.

It is very likely that Bürgi and Napier were unaware of each other's contributions to this area of mathematics. See [20], p. 103-110 for a comprehensive discussion on this priority issue.

So, Bürgi assumed the following geometric sequence, the terms of which he called *black numbers*:

$$b_n = 10^8 \left(1 + \frac{1}{10^4}\right)^n ; n = 0,1,2,... \qquad (4.2.1)$$

The common ratio $\left(1 + \frac{1}{10^4}\right)$ is similar to the common ratio $\left(1 - \frac{1}{10^7}\right)$ of Napier's basal geometric sequence, but coarser. The geometric sequence in Formula (4.2.1) is his discrete classifier of the continuum of positive real numbers. Bürgi, using the same kind of recursion that Napier used in Formula (4.1.4), calculated more than 23,000 terms, covering one decade of numbers.

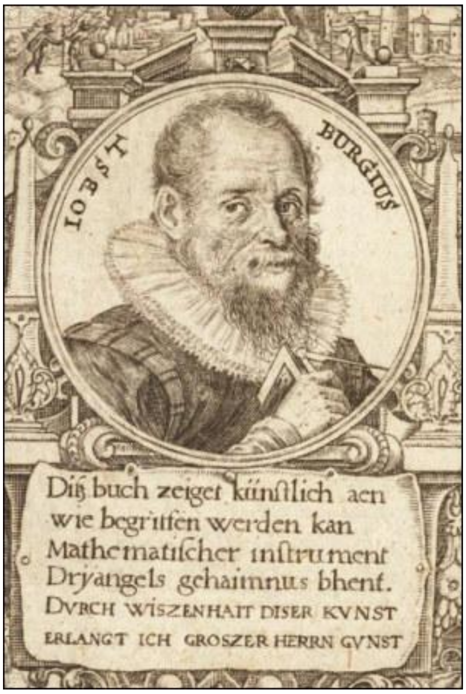

Fig. 4.2.1. Jost Bürgi (1552–1632) was a Swiss astronomer, mathematician, and skilled maker of instruments and clocks. Known in some circles as Justus Byrgius, he worked alongside notable figures such as Tycho Brahe and Johannes Kepler in Kassel and Prague. Remarkably, between approximately 1590 and 1600, Bürgi independently discovered the concept of logarithms, parallel to John Napier's earlier findings.

For his *red numbers r*, he defined the quotient $\frac{r}{10} = n$. Formula (4.2.1) for his *black numbers* can be rewritten as:

$$b = 10^8 \left(1 + \frac{1}{10^4}\right)^{r/10} \qquad (4.2.2)$$

Hence, in modern mathematical notation his red logarithms *r* as a function of his black numbers *b* are:

$$r = 10 \left(\frac{\ln \frac{b}{10^8}}{\ln\left(1 + \frac{1}{10^4}\right)} \right) \qquad (4.2.3)$$

See also Roegel (2010), [12], p. 12. See Figure 4.2.2.

The reason behind Bürgi's introduction of the quotient $r/10$ remains unclear. See for a discussion Truemper (2024), [20], p. 51.

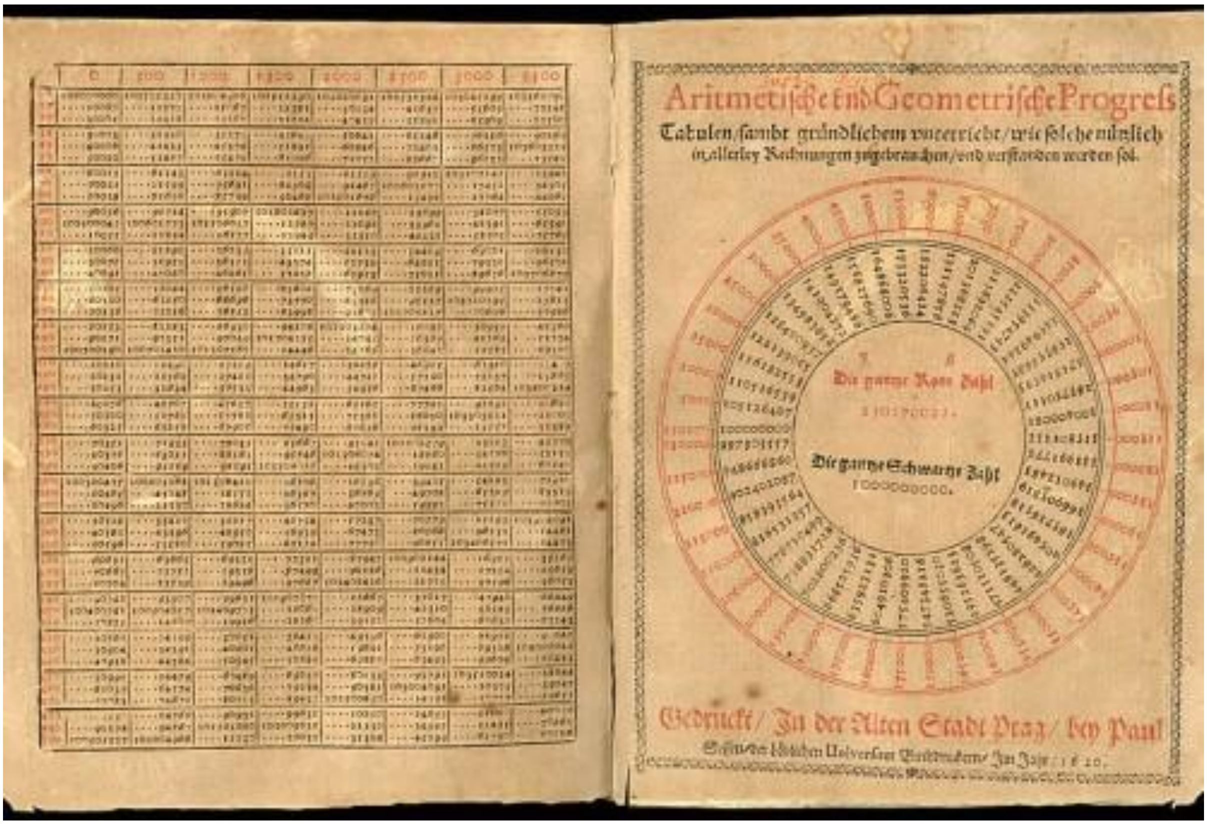

Fig. 4.2.2. The front page of Bürgi's antilogarithm table, alongside a page that highlights logarithms in red and antilogarithms in black. Notably, the Bürgi constant (23027.0022) is prominently marked in red on the front page. It's worth mentioning that a round symbol appears above the first decimal figure, instead of the conventional decimal point separating the integer and decimal parts of the constant.

Despite Kepler's persistent encouragement and his substantial contributions to the manuscript, Bürgi published his table only in 1620 — six years after the release of the *Descriptio* — without clear printed instructions or examples, leaving many users perplexed about its proper application. See Truemper (2024), [20], p. 43. Moreover, using Bürgi's table proved to be more challenging compared to calculating with Napier's table. Refer to Havil, (2014), [2], pp. 264-269). Recognizing its limitations, Kepler ultimately favoured Napier's logarithm table over Bürgi's. See Havil (2014), [2], p. 268.

Ultimately, Briggs' moderner conception of logarithms — which establishes that the logarithm of 1 is 0 and the logarithm of base 10 is 1 — emerged as an evolution from Napier's work, not from Bürgi's. This shift of Briggs made calculations significantly easier, rendering both Napier's and Bürgi's logarithm tables largely obsolete.

CHAPTER 5
NAPIER'S REFERENCE TABLES AND RADICAL MATRIX

5.1. The basal geometric sequence and its arithmetic logarithms

Summarising the above, Napier approached the continuous values $x = \mathrm{Sin}\theta = R\sin\theta$ in the interval OR, on the upper line in Figure 3.2.1, using the basal geometric sequence $\{x_n\}$, with terms $x_n = R\left(1-\frac{1}{R}\right)^n$, as a discrete, numerical approximation. The exact, continuous logarithms $L(x_n) = R\ln\left(\frac{R}{x_n}\right)$ of those geometric terms x_n should ideally be identified with the arithmetic sequence of exponents *n* (the magnitudes) of the common ratio $\rho_1 = \left(1-\frac{1}{R}\right)$, but that unfortunately turns out not to be the case. See for instance Formula (4.1.5).

After all, we found the following discretization of the continuous model:

$$x_n = R\left(1-\frac{1}{R}\right)^n \leftrightarrow L(x_n) \approx \lambda_n = n\left(1+\frac{1}{2R}\right); n = 0,1,2,... \qquad (5.1.1)$$

The exact continuous Napier logarithms do not fully align with arithmetic logarithms, and it's important to note that the arithmetic logarithms do not yield integer values.

Using Formula (5.1.1), an extensive number of arithmetic logarithms — approximately 23 million for the geometric terms in the first decade to the left of the initial point R — can be computed. However, only a finite subset of these, specifically 5,400 (or 5,401 when including 0), is necessary to construct a trigonometric logarithm table.

The challenge lies in selecting the appropriate logarithms. Napier approached this by utilizing multiple geometric sequences across various reference tables. He initially created two tables, which we will refer to as *Reference Tables* T1 and T2, followed by a matrix M, known as the *Radical Matrix*, containing reference values (see Macdonald, 1889, pp. 12-14). In Chapter 5, we will reconstruct these reference tables and the matrix, and in Chapter 6, we will ultimately use them to create Napier's trigonometric logarithm table.

5.2. The primary Reference Table T1

The primary Reference Table T1 contains the first 101 terms x_n of the basal geometric sequence $\{x_n\}$:

$$\mathrm{T}_{1,n} = x_n = R\left(1-\frac{1}{R}\right)^n ; n = 0,1,2,...,100 \qquad (5.2.1)$$

See Table 5.2.1. See also Macdonald (1889), [1] p. 13.

The values presented in Table 5.2.1 were calculated using the straightforward recursive Formula (4.1.4):

$$x_n = \left(1-\frac{1}{R}\right)x_{n-1};\ x_0 = R = 10^7 \qquad (5.2.2)$$

Note, the 101st term (with rank n = 100) in Napier's own primary Reference Table T1 ends with the numerals 50, instead of 51! This is only one of the minor differences between his and our table. See Macdonald (1889), [1], p. 13.

n	Geometrical term	n	Geometrical term	n	Geometrical term	n	Geometrical term	n	Geometrical term
0	10000000.0000000	20	9999980.0000190	40	9999960.0000780	60	9999940.0001770	80	9999920.0003160
1	9999999.0000000	21	9999979.0000210	41	9999959.0000820	61	9999939.0001830	81	9999919.0003240
2	9999998.0000001	22	9999978.0000231	42	9999958.0000861	62	9999938.0001891	82	9999918.0003321
3	9999997.0000003	23	9999977.0000253	43	9999957.0000903	63	9999937.0001953	83	9999917.0003403
4	9999996.0000006	24	9999976.0000276	44	9999956.0000946	64	9999936.0002016	84	9999916.0003486
5	9999995.0000010	25	9999975.0000300	45	9999955.0000990	65	9999935.0002080	85	9999915.0003570
6	9999994.0000015	26	9999974.0000325	46	9999954.0001035	66	9999934.0002145	86	9999914.0003655
7	9999993.0000021	27	9999973.0000351	47	9999953.0001081	67	9999933.0002211	87	9999913.0003741
8	9999992.0000028	28	9999972.0000378	48	9999952.0001128	68	9999932.0002278	88	9999912.0003828
9	9999991.0000036	29	9999971.0000406	49	9999951.0001176	69	9999931.0002346	89	9999911.0003917
10	9999990.0000045	30	9999970.0000435	50	9999950.0001225	70	9999930.0002415	90	9999910.0004006
11	9999989.0000055	31	9999969.0000465	51	9999949.0001275	71	9999929.0002485	91	9999909.0004096
12	9999988.0000066	32	9999968.0000496	52	9999948.0001326	72	9999928.0002556	92	9999908.0004187
13	9999987.0000078	33	9999967.0000528	53	9999947.0001378	73	9999927.0002628	93	9999907.0004279
14	9999986.0000091	34	9999966.0000561	54	9999946.0001431	74	9999926.0002701	94	9999906.0004372
15	9999985.0000105	35	9999965.0000595	55	9999945.0001485	75	9999925.0002775	95	9999905.0004466
16	9999984.0000120	36	9999964.0000630	56	9999944.0001540	76	9999924.0002850	96	9999904.0004561
17	9999983.0000136	37	9999963.0000666	57	9999943.0001596	77	9999923.0002926	97	9999903.0004657
18	9999982.0000153	38	9999962.0000703	58	9999942.0001653	78	9999922.0003003	98	9999902.0004754
19	9999981.0000171	39	9999961.0000741	59	9999941.0001711	79	9999921.0003081	99	9999901.0004852
20	9999980.0000190	40	9999960.0000780	60	9999940.0001770	80	9999920.0003160	100	**9999900.0004951**

Table 5.2.1. The first 101 terms of the basal geometric sequence from the primary Reference Table T1, computed using recursion in Excel. Each term is displayed with a precision of seven digits before and after the decimal point.

We calculate the geometric terms of Formula (5.2.2) using a recursive method that maintains 14 digits of precision — 7 digits to the left and 7 to the right of the decimal point — reflecting Napier's original technique.

It is essential to emphasize that these geometric terms are relevant only to a narrow segment of the interval OR, particularly in the immediate vicinity of the starting point R on the upper line of the kinematic model.

Napier reasoned further, that the approximate arithmetic logarithms λ_n of the x_n-values in Formula (5.2.2) must form an arithmetic sequence $\{\lambda_n\}$. We know the exact starting logarithm $L(x_0) = L(R) = 0 = \lambda_0$, and Formula (4.1.8) says $L(x_1) \approx \left(1 + \dfrac{1}{2R}\right) = \lambda_1 = 1.000\,000\,05$. Hence, this value must be also the common difference $\delta_1 = \left(1 + \dfrac{1}{2R}\right)$ of the arithmetic sequence $\{\lambda_n\}$ of the approximate arithmetic logarithms λ_n.

This enables the calculation of all 101 arithmetic logarithms λ_n in Reference Table T1 to an accuracy of eight decimal places with relative ease. Table 5.2.2 presents these values $\lambda_n = 0 + n\delta_1; n = 0,1,2,...,100$. To maintain brevity, we have included only a selection of the most significant values in the table.

Column 1 of Table 5.2.2 contains the exponents n of the terms of the basal geometric sequence $\{x_n\}$ with initial term R. Of course, those are also ranking numbers. Column 2 contains the first 101 terms of the basal geometric sequence $\{x_n\}$; values in this column are copied from Table 5.2.1. Column 3 contains the terms of the sequence $\lambda_n = 0 + n\delta_1$ of arithmetic logarithms that are *approximations* of the corresponding exact logarithms $L(x_n)$. As said, these arithmetic logarithms λ_n form an arithmetic sequence $\{\lambda_n\}$ with initial term 0 and common difference $\delta_1 = \left(1 + \frac{1}{2R}\right) = 1.000\ 000\ 05$.

n	Geometrical term	Arithm. Log	Exact Log
0	10000000.0000000	0.00000000	0.00000000
1	9999999.0000000	1.00000005	1.00000005
2	9999998.0000001	2.00000010	2.00000010
3	9999997.0000003	3.00000015	3.00000015
4	9999996.0000006	4.00000020	4.00000020
5	9999995.0000010	5.00000025	5.00000025
6	9999994.0000015	6.00000030	6.00000030
7	9999993.0000021	7.00000035	7.00000035
8	9999992.0000028	8.00000040	8.00000040
9	9999991.0000036	9.00000045	9.00000044
10	9999990.0000045	10.00000050	10.00000050
...			
...			
95	9999905.0004466	95.00000475	95.00000470
96	9999904.0004561	96.00000480	96.00000475
97	9999903.0004657	97.00000485	97.00000480
98	9999902.0004754	98.00000490	98.00000485
99	9999901.0004852	99.00000495	99.00000490
100	9999900.0004951	100.00000500	100.00000495

Table 5.2.2. First 11 and last 6 entries of Napier's complete primary Reference Table T1, with 101 geometric terms x_n, as well as their approximate arithmetic logarithms λ_n in the third column. Those logarithms we compare with the exact logarithms $L(x_n)$ from the kinematic model, in the fourth column.

This difference also represents the arithmetic logarithm λ_1 of the first geometric term that is not equal to value *R*. Refer to Macdonald (1889), [1], pp. 22–23.

Column 4 presents the exact Napier logarithms, calculated using Excel with Formula (3.2.6) from the continuous kinematic model. This column serves solely for comparison with the third column and does not belong to Napier's original Reference Table T1. For a detailed comparison, please see Table 5.2.2 alongside Table 4.9 in Havil (2014), [2], p. 112.

Remarkably, the approximate arithmetic logarithms calculated by Napier, as shown in the fourth column of Table 5.2.2, differ very little from the exact logarithms of the continuous model — a model of which Napier was unaware. The values agree within an impressive margin of 8 or 9 significant figures. Achieving such accuracy using only pen and paper must have posed a considerable challenge for Napier.

Furthermore, both column 3 (approximate arithmetic logarithms) and column 4 (exact Napier logarithms) demonstrate slight discrepancies from the non-negative integers *n*, consistent with our expectations based on Formula (5.1.1).

In summary, the following Formula applies to the primary Reference Table T1:

$$\begin{cases} T_{1,n} = x_n = R\left(1-\frac{1}{R}\right)^n \\ \lambda_n = 0 + n\left(1+\frac{1}{2R}\right); n = 0,1,...,100 \end{cases} \tag{5.2.3}$$

The final geometric term $R\left(1-\frac{1}{R}\right)^{100}$ in Reference Table T1 serves as a reference for both Reference Table T2 and the Radical Matrix M.

It's important to note that the first 101 geometric terms correspond to Sines of angles that are all quite close to 90°.

5.3. The secondary Reference Table T2

As point X moves leftward along the upper line of the kinematic model, it progresses from its starting position R towards the origin O in a stepwise manner, following the terms of the basal geometric sequence $\{x_n\}$. The size of each step taken by X is minuscule relative to the length of the interval OR. Consequently, the 101 terms x_n listed in Reference Table T1 address only a tiny fraction of the interval in the close proximity to point R.

Napier considered expediting this process by introducing a second geometric sequence $\{x_m\}$ (mind: with rank *m* not *n*!) with a larger gap between each consecutive term. What specific sequence should he choose for this purpose?

5.3.1. A second geometric sequence

Napier uncovered that new sequence with a distinct common ratio by making an unexpected approximation of the last term in Reference Table T1. The final geometric term in Reference Table T1 is:

$$T_{1,100} = R\left(1-\frac{1}{R}\right)^{100} = 9\,999\,900{,}000\,495\,1 \approx \\ \approx R\left(1-\frac{100}{R}\right) = 10^7\left(1-\frac{1}{10^5}\right) = 9\,999\,900{,}000\,000\,0 \tag{5.3.1}$$

The term $R\left(1-\frac{100}{R}\right)$ we identify here corresponds to the first-order Maclaurin[36] expansion of the geometric term $R\left(1-\frac{1}{R}\right)^{100}$. However, it is important to note that Napier was unaware of this expansion and solely found the new sequence to be practical for his needs.

From the term $R\left(1-\frac{100}{R}\right)$ a second common ratio $\rho_2 = \left(1-\frac{1}{10^5}\right)$ of a second geometric sequence $\{x_m\}$ appears.

As point X moves gradually and systematically to the left through the interval OR, it ad-

[36] Colin Maclaurin was a Scottish mathematician who lived from 1698 to 1746. He is best known for his work in calculus and for the Maclaurin series, which is a special case of the Taylor series.

vances using the terms of the second geometric sequence. Each of these steps is 100 times larger than the steps taken using the geometric sequence in the primary Reference Table T1. As a result, Napier created a secondary Reference Table T2, consisting of 51 terms:

$$T_{2,m} = x_m = R\left(1 - \frac{1}{10^5}\right)^m ; m = 0,1,2,...,50 \tag{5.3.2}$$

Table 5.3.1 shows the terms of that second geometric sequence. See also Macdonald (1889), [1], pp. 13 – 14[37] .

m	Geometrical term	*m*	Geometrical term	*m*	Geometrical term
0	10000000.000000	**20**	9998000.189989	**40**	9996000.779901
1	9999900.000000	**21**	9997900.209987	**41**	9995900.819893
2	9999800.001000	**22**	9997800.230985	**42**	9995800.860885
3	9999700.003000	**23**	9997700.252982	**43**	9995700.902877
4	9999600.006000	**24**	9997600.275980	**44**	9995600.945868
5	9999500.010000	**25**	9997500.299977	**45**	9995500.989858
6	9999400.015000	**26**	9997400.324974	**46**	9995401.034848
7	9999300.021000	**27**	9997300.350971	**47**	9995301.080838
8	9999200.027999	**28**	9997200.377967	**48**	9995201.127827
9	9999100.035999	**29**	9997100.405963	**49**	9995101.175816
10	9999000.044999	**30**	9997000.434959	**50**	9995001.224804
11	9998900.054998	**31**	9996900.464955		
12	9998800.065998	**32**	9996800.495950		
13	9998700.077997	**33**	9996700.527945		
14	9998600.090996	**34**	9996600.560940		
15	9998500.104995	**35**	9996500.594935		
16	9998400.119994	**36**	9996400.629929		
17	9998300.135993	**37**	9996300.665922		
18	9998200.152992	**38**	9996200.702916		
19	9998100.170990	**39**	9996100.740909		
20	9998000.189989	**40**	9996000.779901		

Table 5.3.1. Partial (still incomplete) secondary Reference Table T2, with the first 51 terms of the geometric sequence $\{x_m\}$, with common ratio $\rho_2 = 1 - \frac{1}{10^5}$, and initial term R, calculated with recursion in Excel. See [1], p. 14. Note: 'only' 6 digits after the decimal point.

The second term $T_{2,1}$ (rank *m* = 1) of the second geometric sequence $\{x_m\}$ is slightly smaller than the last term $T_{1,100}$ in the Reference Table T1. The term $T_{2,1}$ is thus located in the interval OR somewhat left of $T_{1,100}$. The term $T_{2,1}$ jumps over $T_{1,100}$; the arithmetic logarithm of $T_{2.1}$ must therefore be somewhat larger than that of $T_{1.100}$. That is the case, indeed. See the 3rd column of Table 5.3.2.

[37] Napier gives 9 995 001.222 927 as the last term of table 5.3.1, but that value is incorrect, as noted by Macdonald, who apparently redid Napier's calculation. See Macdonald (1889) [1], p. 14.

	Geometrical term	Arithmetical Log.	Exact Log.
$T_{1,100}$ =	9 999 900.000 495 1	100.000 005 00	100.000 004 95
$T_{2,1}$ =	9 999 900.000 000 0	100.000 500 0	100.000 500 0

Table 5.3.2. Comparison of the last row of the primary Reference Table T1 (n = 100) with the second row of the secondary Reference Table T2 (m = 1). It shows term $T_{2,1}$ is somewhat smaller than $T_{1,00}$; the logarithm of $T_{2,1}$ is therefore somewhat larger than the logarithm of $T_{1,100}$.

5.3.2. A second sequence of arithmetic logarithms

The arithmetic logarithms of the geometric terms $T_{2,m}$ must also form an arithmetic sequence. Napier firstly calculated the logarithm of $T_{2.1}$ = 9 999 900.000 000 0.

m	Geometrical Term	Arithmetical Log.	Exact Nap. Log.
0	10000000.0000000	0.0000000	0.0000000
1	9999900.0000000	100.0005000	100.0005000
2	9999800.0010000	200.0010000	200.0010000
3	9999700.0030000	300.0015000	300.0015000
4	9999600.0060000	400.0020000	400.0020000
5	9999500.0099999	500.0025000	500.0025000
…	…	…	…
47	9995301.0808379	4700.0235000	4700.0235001
48	9995201.1278271	4800.0240000	4800.0240001
49	9995101.1758158	4900.0245000	4900.0245001
50	9995001.2248040	5000.0250000	5000.0250001

Table 5.3.3. Part of the complete secondary Reference Table T2, showing the first 6 and last 4 terms of the second geometric sequence $\{x_m\}$ with common ratio $\rho_2 = 1 - \frac{1}{10^5}$ and initial term R. The approximate logarithms in column 3 form an arithmetic sequence. Column 4 contains the exact Napier logarithms of the geometric terms in column 2, calculated with recursion in Excel. See also Havil (2014), [2], p. 113.

This term is slightly smaller than $T_{1,100}$ = 9 999 900.000 495 1, of which the arithmetic logarithm is $\lambda(T_{1,100})$ = 100.000 005 00. See again Table 5.3.2.

We compare this last arithmetic logarithm of Table 5.3.1 with the one we need for $T_{2,1}$ by applying the inequality in Formula (3.5.6):

$$R \cdot \frac{T_{1,100} - T_{2,1}}{T_{1,100}} < L(T_{2,1}) - L(T_{1,100}) < R \cdot \frac{T_{1,100} - T_{2,1}}{T_{2,1}} \tag{5.3.3}$$

Napier finds from this inequality $\lambda\left(T_{2,1}\right) \approx L(T_{2,1}) = 100.000\ 500\ 0$, in seven decimals. See for the somewhat laborious derivation of this value, Havil (2014), [2], pp. 112-113.

This arithmetic logarithm must also be the common difference δ_2 of the arithmetic sequence of the logarithms in Reference Table T2. Thus, $\delta_2 = 100.000500\ 0$ and the second arithmetic sequence of logarithms becomes $\lambda_m = 0 + m\delta_2$

In summary, for the secondary Reference Table T2 we find:

$$\begin{cases} T_{2,m} = x_m = 10^7\left(1-\frac{1}{10^5}\right)^m \leftrightarrow \lambda_m = 0 + m\delta_2\,;\, m = 0,1,\ldots,50 \\ \delta_2 = 100.000\,500\,0 \end{cases} \tag{5.3.4}$$

T2 is the second Reference Table required for constructing the Radical Matrix M detailed below. Similar to Table T1, the geometric terms x_m in T2 represent the Sines of angles that are very close to 90°. In fact, when expressed to seven significant figures, the final term reveals that the geometric terms of Reference Table T2 cover less than 2 arc degrees. Consequently, the terms in the second geometric sequence $\{x_m\}$ do not venture far leftward through the interval OR. The extended Radical Matrix M, which we will discuss in the next Paragraph 5.4), enabled Napier to effectively address much smaller angles, down to 30°.

5.4. The double-geometric terms in the Radical Matrix M

		q →		
		0	1	2
	0	10000000.000000	9900000.000000	9801000.000000
	1	9995000.000000	9895050.000000	9796099.500000
	2	9990002.500000	9890102.475000	9791201.450250
	3	9985007.498750	9885157.423763	9786305.849525
	4	9980014.995001	9880214.845051	9781412.696600
	5	9975024.987503	9875274.737628	9776521.990252
	6	9970037.475009	9870337.100259	9771633.729257
	7	9965052.456272	9865401.931709	9766747.912392
p	8	9960069.930044	9860469.230743	9761864.538436
↓	9	9955089.895079	9855538.996128	9756983.606167
	10	9950112.350131	9850611.226630	9752105.114364
	11	9945137.293956	9845685.921017	9747229.061806
	12	9940164.725309	9840763.078056	9742355.447275
	13	9935194.642946	9835842.696517	9737484.269552
	14	9930227.045625	9830924.775169	9732615.527417
	15	9925261.932102	9826009.312781	9727749.219653
	16	9920299.301136	9821096.308125	9722885.345044
	17	9915339.151486	9816185.759971	9718023.902371
	18	9910381.481910	9811277.667091	9713164.890420
	19	9905426.291169	9806372.028257	9708308.307975
	20	9900473.578023	9801468.842243	9703454.153821

Table 5.4.1. Sub-table of matrix M. The first three incomplete, columns of the 21 x 69 Radical Matrix M with just the geometric terms. The index counter p goes vertically through the row numbers and index counter q goes horizontally through the column numbers. Table generated by Excel.

The Radical Matrix M serves as the central logarithm table that Napier utilized to compute the logarithms of trigonometric quantities. For a closer look, refer to Tables 5.4.1 and 5.4.2, which are sub-tables derived from M, focusing specifically on geometric terms. The complete Radical Matrix M comprises 69 consecutive columns of decreasing geometric terms $x_{p,q}$ alongside their corresponding increasing logarithms $\lambda_{p,q}$ — approximations that we will calculate in the upcoming Paragraph 5.5. For further details, consult Macdonald (1889), [1], pp. 15-16.

Additionally, the primary and secondary Reference Tables T1 and T2 provide the essential reference values that Napier needed to establish the initial logarithms of the geometric terms ranked at p = 1 in columns 0 and 1 of Matrix M[38]. Please refer to Table 5.4.1 for this information.

Notice, the last term of Reference Table T2, $T_{2,50} = 10^7\left(1-\frac{1}{10^5}\right)^{50} = 9\,995\,001{,}224\,804\,0$, is slightly larger than its first order Maclaurin approximation:
$10^7\left(1-\frac{50}{10^5}\right) = 10^7\left(1-\frac{1}{2000}\right) = 9\,995\,000{,}000\,000\,0$. See the geometric value $x_{1,0}$ in Table 5.4.1.

This led Napier to propose the following idea: a third geometric sequence with a common ratio of:

$$\rho_3 = 1-\frac{1}{2000} \tag{5.4.1}$$

passes through the interval OR even faster than the geometric sequence with the second common ratio $\rho_2 = 1-\frac{1}{10^5}$ of Reference Table T2.

Following the same line of thought, Napier found even a fourth common ratio of a geometric sequence at his disposal:

$$\rho_4 = \left(1-\frac{1}{2000}\right)^{20} = 0.9900474\ldots \approx 1-\frac{1}{100} = 0.9900000 \tag{5.4.2}$$

If we multiply this fourth common ratio with $R = 10^7$, then we find the geometric value $x_{0,1}$ in matrix M. Refer to Table 5.4.1. We have identified the two non-trivial initial values for the first column of matrix M, as well as for the first row.

So, the Radical Matrix M contains specific *x*-values (Sines) in the interval OR: each column *q* of M, with index counter $q = 0,1,2,\ldots,68$, contains 21 terms of a geometric sequence with common ratio $\rho_3 = 1-\frac{1}{2000}$; each row *p*, with index counter $p = 0,1,2,\ldots,20$, of M contains 69 terms of a geometric sequence with common ratio $\rho_4 = 1-\frac{1}{100}$. See Table 5.4.2. As a result, the last geometric term in a column of the Radical Matrix M, with rank *p* = 20, is somewhat larger than the first geometric term, with rank *p* = 0, in the subsequent column of M. This observation allows us to conceptualize the 69 columns, arranged vertically, as a single, cohesive column of terms.

The initial term of the matrix M is of course $x_{0,0} = R = 10^7$. So, in one Formula, the double-geometric terms in matrix M are:

$$x_{p,q} = R\left(1-\frac{1}{2000}\right)^p\left(1-\frac{1}{100}\right)^q;\ p = 0,1,2,3,\ldots,20;\ q = 0,1,2,3,\ldots,68 \tag{5.4.3}$$

38 Unlike Napier, we do not number the rows and columns from 1, but from rank number 0.

		q →		
		66	67	68
	0	5151371.174238	5099857.462496	5048858.887871
	1	5148795.488651	5097307.533764	5046334.458427
	2	5146221.090907	5094758.879998	5043811.291198
	3	5143647.980361	5092211.500558	5041289.385552
	4	5141076.156371	5089665.394807	5038768.740859
	5	5138505.618293	5087120.562110	5036249.356489
	6	5135936.365484	5084577.001829	5033731.231811
	7	5133368.397301	5082034.713328	5031214.366195
p	8	5130801.713102	5079493.695971	5028698.759012
↓	9	5128236.312246	5076953.949123	5026184.409632
	10	5125672.194090	5074415.472149	5023671.317427
	11	5123109.357993	5071878.264413	5021159.481768
	12	5120547.803314	5069342.325280	5018648.902028
	13	5117987.529412	5066807.654118	5016139.577577
	14	5115428.535647	5064274.250291	5013631.507788
	15	5112870.821379	5061742.113166	5011124.692034
	16	5110314.385969	5059211.242109	5008619.129688
	17	5107759.228776	5056681.636488	5006114.820123
	18	5105205.349161	5054153.295670	5003611.762713
	19	5102652.746487	5051626.219022	5001109.956832
	20	5100101.420113	5049100.405912	4998609.401853

Table 5.4.2. Sub-table of matrix M. The last three incomplete columns of the 21 x 69 radical matrix M just with geometric terms. Table generated by Excel.

If one traverses matrix M vertically down a column, we take finer steps to the left within the interval OR, utilizing the common ratio $\rho_3 = 1 - \frac{1}{2000}$. Conversely, when we move horizontally across a row of M, our steps to the left within the interval OR are coarser, governed by the common ratio $\rho_4 = 1 - \frac{1}{100}$. This method enables us to access 1,449 consecutive double-geometric terms, along with their approximate arithmetic logarithms, as will be discussed in the next Paragraph 5.4.

Ultimately, the values of the double-geometric terms $x_{p,q}$ in the Radical Matrix M run leftwards in the interval OR from $R = 10^7$ (farthest left top in matrix M, see Table 5.4.1) to approximately to the half-way value $H = \frac{1}{2}R = 5 \cdot 10^6 = \text{Sin}30°$ (farthest far right bottom in matrix M, see Table 5.4.2). Note that these values correspond to the rightmost decade (10^6, 10^7), to the left of the initial point R of interval OR. Refer to Figure 3.3.1 for clarification.

5.5. The logarithms of the 1st column geometric terms of the Radical Matrix M

It is helpful to label the columns of the radical matrix M using a numerical system: M0, M1, M2, and so forth. In the leftmost column, M0, of Table 5.4.1, we find the initial geometric term $x_{0,0} = R = 10^7$, which has an arithmetic logarithm of 0.0000000.

The logarithms of the geometric terms in column M0 must form an arithmetic sequence $\{\lambda_{p,0}\}$, with the initial term *R*, and with a specific *vertical* common difference δ_3, which is also the arithmetic logarithm $\lambda_{1,0}$ of the earlier calculated geometric term $x_{1,0}$.

	Column M0 of matrix M	
p	**Geom. Term**	**Aritm. Log.**
0	10000000.0000000	0.0000000
1	9995000.0000000	5001.250
2	9990002.5000000	10002.50
3	9985007.4987500	15003.75
4	9980014.9950006	20005.00
5	9975024.9875031	25006.25
6	9970037.4750094	30007.50
7	9965052.4562719	35008.75
8	9960069.9300437	40010.00
9	9955089.8950787	45011.25
10	9950112.3501312	50012.50
11	9945137.2939561	55013.75
12	9940164.7253091	60015.00
13	9935194.6429465	65016.25
14	9930227.0456250	70017.50
15	9925261.9321022	75018.75
16	9920299.3011362	80020.00
17	9915339.1514856	85021.25
18	9910381.4819098	90022.50
19	9905426.2911689	95023.75
20	9900473.5780233	100025.0

Table 5.5.1. Complete column M_0 of the Radical Matrix M. Here the geometric terms are supplemented with their arithmetic logarithms, expressed in 7 significant figures. Table generated by Excel.

First, we determine the arithmetic logarithm of the second geometric term in M0 (with rank *p* = 1), that is $x_{1,0} = R\left(1 - \frac{1}{2000}\right)^1 = 9\ 995\ 000{,}000\ 000\ 0$.

We calculate the logarithm of that term via the last (slightly larger) geometric term in Reference Table T2, namely $\mathrm{T}_{2,50} = 9\ 995\ 001{,}224\ 804\ 0$, with the already calculated arithmetic logarithm $\lambda_{\mathrm{T}_{2,50}} = 5\ 000{,}025\ 000\ 0$. We again apply the inequality in Formula (3.5.6):

$$R \cdot \frac{\mathrm{T}_{2,50} - x_{1,0}}{\mathrm{T}_{2,50}} < \lambda_{1,0} - \lambda_{\mathrm{T}_{2,50}} < R \cdot \frac{\mathrm{T}_{2,50} - x_{1,0}}{x_{1,0}} \tag{5.5.1}$$

From this follows the arithmetic logarithm $\lambda_{1,0} = 5\ 001.250\ 416\ 8 \approx 5\ 001.250$ in 7 significant figures. This logarithm represents the vertical common difference δ_3 of the arithmetic logarithms found in column M0. Using this difference, the other arithmetic logarithms of the geometric terms in the leftmost column (M0) of matrix M can be calculated with relative ease.

For reference, consult Table 5.5.1, which features the now-complete column M0 of the Radical Matrix M. Additionally, refer to Macdonald (1889) [1], p. 34 and Havil (2004), [2], pp. 113-114 for further information.

For comparison with the relevant values of the Primary and Secondary Reference Tables T1 and T2, see Table 5.5.2.

	Geometrical term	Arithmetical Log.	Exact Napier Log.
$T_{1,100}$:	9 999 900.000 495 0	100.000 005 0	100.000 004 95
$T_{2,0}$:	9 999 900.000 000 0	100.000 500 0	100.000 500 0
$T_{2,50}$:	9 995 001.224 804 0	5 000.025 000 0	5 000.025 000 1
$x_{1,0}$:	9 995 000.000 000 0	5 001.250 416 8 = δ_3	

Table 5.5.2. Comparison of the last values of Reference Tables T1 and T2 and the values on the second row (p = 1) of the leftmost column M_0 of the Radical Matrix M.

5.6. The logarithm of the 1st term of the 2nd column of matrix M

The arithmetic logarithm of $x_{0,1} = 9\ 900\ 000.000\ 000$, which is the top geometric term of the second column M1 of Radical Matrix M, must correspond to the arithmetic logarithm 100025.0 of the last geometric term $x_{20,0} = 9\ 900\ 473.578\ 023\ 3$ in the first column M0 of M.

The arithmetic logarithms of the first row of M constitute an arithmetic sequence with *horizontal*, common difference δ_4, which also equals the arithmetic logarithm $\lambda_{0,1} = \lambda(9\,900\,000.000\,000)$. This logarithm one can calculate via the last logarithm of column M0, that is $\lambda_{20,0} = 100\ 025.0$, as follows. Napier realized that the vertical ratio $\frac{x_{0,1}}{x_{20,0}}$ must be equal to a horizontal ratio $\frac{x}{R}$, for some *x*-value yet to be calculated, and hence, that both ratios must have the same logarithm. We find for *x*:

$$x = R\frac{x_{0,1}}{x_{20,0}} = 9\ 999\ 521.661\ 242\ 20 \tag{5.6.1}$$

The logarithm of a ratio is the difference of the logarithms:

$$\lambda_{0,1} - \lambda_{20,0} = L(x) - L(R) \tag{5.6.2}$$

from which follows:

$$\lambda_{0,1} = \lambda_{20,0} + L(x) \tag{5.6.3}$$

From the inequality (3.5.2) for one logarithm, $X = \frac{R}{x} \to R - x < L(x) < \frac{R}{x}(R - x)$, follows:

$$478.338\ 757\ 8 < L(x) < 478.361\ 639\ 7 \tag{5.6.4}$$

Because $\lambda_{20,0} = 100025.00$, and by using Formulas (5.6.3) and (5.6.4), we arrive at:

$$100\ 503.338\ 757\ 8 < \lambda_{0,1} < 100\ 503.361\ 639\ 7 \tag{5.6.5}$$

Napier found, in 7 significant figures, as the *horizontal* common difference:

$\delta_4 = \lambda_{0,1} = 100\,503.358\,530\,1 \approx 100\,503.4$. See Havil (2004), [2], pp. 115-116.

Table 5.6.1 provides an overview of several values derived from the aforementioned reasoning.

	Geometrical term	Arithmetical Log.
$T_{1,100}$:	9 999 900.000 495 1	100.000 005 00
$T_{2,0}$:	9 999 900.000 000 0	100.000 500 0
$T_{2,50}$:	9 995 001.224 804 0	5 000.025 000 0
$x_{1,0}$:	9 995 000.000 000 0	$5\,001.250\,416\,8 = \delta_3$ (vertical)
$x_{20,0}$:	9 900 473.578 023 3	100 025.0
$x_{0,1}$:	9 900 000.000 000	$100\,503.4 = \delta_4$ (horizontal)

Table 5.6.1. Comparison of the relevant values of T1 and T2, and three of the most characteristic values of the Radical Matrix M. The last two numbers rounded to 7 significant figures.

5.7. The Radical Matrix M completed

In summary we have:

- Vertically: the arithmetic logarithms of the geometric terms in each matrix *column* form an arithmetic sequence with *vertical* common difference δ_3 = 5 001.250 416 8.
- Horizontally: the arithmetic logarithms of the geometric terms in each matrix *row* form an arithmetic sequence with *horizontal* common difference δ_4 = 100 503.358 535 01.

	Column M1 of matrix M			Column M2 of matrix M	
p	Geom. Term	Aritm. Log.	p	Geom. Term	Aritm. Log.
0	9900000.000000	100503.4	0	9801000.000000	201006.7
1	9895050.000000	105504.6	1	9796099.500000	206008.0
2	9890102.475000	110505.9	2	9791201.450250	211009.2
3	9885157.423763	115507.1	3	9786305.849525	216010.5
4	9880214.845051	120508.4	4	9781412.696600	221011.7
5	9875274.737628	125509.6	5	9776521.990252	226013.0
6	9870337.100259	130510.9	6	9771633.729257	231014.2
7	9865401.931709	135512.1	7	9766747.912392	236015.5
8	9860469.230743	140513.4	8	9761864.538436	241016.7
9	9855538.996128	145514.6	9	9756983.606167	246018.0
10	9850611.226630	150515.9	10	9752105.114364	251019.2
11	9845685.921017	155517.1	11	9747229.061806	256020.5
12	9840763.078056	160518.4	12	9742355.447275	261021.7
13	9835842.696517	165519.6	13	9737484.269552	266023.0
14	9830924.775169	170520.9	14	9732615.527417	271024.2
15	9826009.312781	175522.1	15	9727749.219653	276025.5
16	9821096.308125	180523.4	16	9722885.345044	281026.7
17	9816185.759971	185524.6	17	9718023.902371	286028.0
18	9811277.667091	190525.9	18	9713164.890420	291029.2
19	9806372.028257	195527.1	19	9708308.307975	296030.5
20	9801468.842243	200528.4	20	9703454.153821	301031.7

Table 5.7.1. 2nd and 3rd column of the Radical Matrix M, with arithmetic logarithms, rounded to 1 decimal place. Generated by Excel. See also Havil (2004), [2], p. 117.

Keeping this in mind, constructing the remaining columns of the Radical Matrix M is relatively straightforward. For example, refer to the Excel-generated columns M1 and M2, along with columns M67 and M68, which can be found in Tables 5.7.1 and 5.7.2.

	Column M67 of matrix M			Column M68 of matrix M	
p	**Geom. Term**	**Aritm. Log.**	**p**	**Geom. Term**	**Aritm. Log.**
0	5099857.462496	6733725.0	**0**	5048858.887871	6834228.4
1	5097307.533764	6738726.3	**1**	5046334.458427	6839229.6
2	5094758.879998	6743727.5	**2**	5043811.291198	6844230.9
3	5092211.500558	6748728.8	**3**	5041289.385552	6849232.1
4	5089665.394807	6753730.0	**4**	5038768.740859	6854233.4
5	5087120.562110	6758731.3	**5**	5036249.356489	6859234.6
6	5084577.001829	6763732.5	**6**	5033731.231811	6864235.9
7	5082034.713328	6768733.8	**7**	5031214.366195	6869237.1
8	5079493.695971	6773735.0	**8**	5028698.759012	6874238.4
9	5076953.949123	6778736.3	**9**	5026184.409632	6879239.6
10	5074415.472149	6783737.5	**10**	5023671.317427	6884240.9
11	5071878.264413	6788738.8	**11**	5021159.481768	6889242.1
12	5069342.325280	6793740.0	**12**	5018648.902028	6894243.4
13	5066807.654118	6798741.3	**13**	5016139.577577	6899244.6
14	5064274.250291	6803742.5	**14**	5013631.507788	6904245.9
15	5061742.113166	6808743.8	**15**	5011124.692034	6909247.1
16	5059211.242109	6813745.0	**16**	5008619.129688	6914248.4
17	5056681.636488	6818746.3	**17**	5006114.820123	6919249.6
18	5054153.295670	6823747.5	**18**	5003611.762713	6924250.9
19	5051626.219022	6828748.8	**19**	5001109.956832	6929252.1
20	5049100.405912	6833750.0	**20**	4998609.401853	6934253.4

Table 5.7.2. The rightmost two columns M67 and M68 of M, with the arithmetic logarithms, rounded to 1 decimal place. Generated by Excel. See also Havil (2004), [2], p. 118.

5.8. The logarithm of *H* = Sin 30°

Notice, the last geometric term in column M68 of the Radical Matrix M is $x_{20,68} = 4998609.401853 \approx 5.00 \cdot 10^6 = H$, which is the Sinus of 30°. See Table 5.7.1.

The point H lies halfway through the interval OR; the geometric term $x_{20,68}$ is somewhat smaller than *H*; the penultimate geometric term $x_{19,68} = 5001109.956832$ is somewhat greater than *H*. Hence, the logarithm of $x_{19,68}$ is smaller than the one of $x_{20,68}$. See Figure 5.7.1. As we shall see in the next Chapter 6, logarithms of *x*-values left of point H can be calculated via the logarithms right of it. That is presumably the reason Napier ended here his Radical Matrix M, and an explanation of the, at first site notable number of 69 columns.

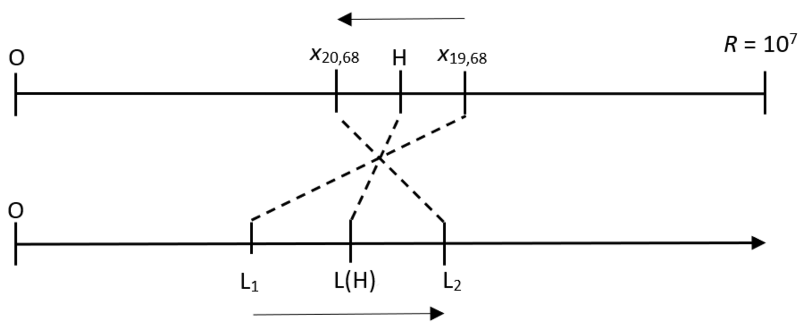

Fig. 5.7.1. The interpolation of the logarithm of H.

To calculate the arithmetic logarithm of *H*, Napier used Formula (3.5.6) again, that is

$x_1 > x_2 \;\to\; R\cdot\frac{x_1 - x_2}{x_1} < L(x_2) - L(x_1) < R\cdot\frac{x_1 - x_2}{x_2}$. With $x_1 = x_{19,68}$ and $x_2 = H$, this inequality goes over into:

$$R\cdot\frac{x_{19,68} - H}{x_{19,68}} < \lambda(H) - \lambda(x_{19,68}) < R\cdot\frac{x_{19,68} - H}{H} \qquad (5.8.1)$$

Or, numerically:

$$\begin{aligned}&2219.42097092 + 6929252.138300 < \lambda(H) < 2219.91366321 + 6929252.138000\\ &\to 6931471.559271 < \lambda(H) < 6931472.051963\end{aligned} \qquad (5.8.2)$$

So, in 7 significant digits:

$$\lambda(H) = 6931472 \qquad (5.8.3)$$

which is indeed the exact Napier logarithm, in 7 significant digits, of the angle of 30°, that our own Excel sheet also produces.

When we use instead the geometric term $x_{20,68} = x_2$ and $x_1 = H$, then by applying Formula (3.5.6), we find:

$$\begin{aligned}&R\cdot\frac{H - x_{20,68}}{H} < \lambda(x_{20,68}) - \lambda(H) < R\cdot\frac{H - x_{20,68}}{x_{20,68}}\\ &6931471.418703 < \lambda(H) < 6931472.192423\\ &\lambda(H) = 6931471.806 = 6931472 \text{ (7 digits)}\end{aligned} \qquad (5.8.4)$$

The arithmetic logarithm of *H* we now calculate, as the average of the two borders of the inequality. In 7 significant digits we arrive at the same value as in Formula (5.8.3):

$$\lambda(\mathrm{Sin}30^\circ) = 6931472 \qquad (5.8.5)$$

CHAPTER 6
CONSTRUCTING THE TRIGONOMETRIC LOGARITHM TABLE

6.1. Calculation of a logarithm via the Radical Matrix M

We are now poised to calculate logarithms using the Radical Matrix M. This raises an interesting question: how did Napier compute the logarithm of a specific Sine value $x = \text{Sin}\,\theta = R\sin\theta$ using this matrix? To explore this, we turn to the example provided in Havil (2014), [2], on pages 119-120.

6.1.1. Example: angle greater than 30°

Table 6.1.1, created using Excel, presents the sine values for angles ranging from 48° to 49°, as well as their complementary angles from 41° to 42°. The table provides both the Sine values and the precise, continuous Napier logarithm, with all figures rounded to seven significant digits, as indicated in the bottom row of the left section. We find $\text{Sin}(48°30') = 7489557$ as well as the exact, continuous Napier logarithm of that Sine: $L(\text{Sin}48°30') = 2\,890754$

Degr	Min	Sin	NapLog	plus/min	NapLog	Sin			Min	Sin	NapLog	plus/min	NapLog	Sin	
48	0	7431448	2968643	-1049117	4017760	6691306	60		30	7489557	2890754	-1224781	4115535	6626200	30
	1	7433394	2966025	-1054967	4020992	6689144	59		31	7491484	2888181	-1230643	4118824	6624022	29
	2	7435340	2963408	-1060817	4024225	6686981	58		32	7493411	2885610	-1236505	4122115	6621842	28
	3	7437285	2960793	-1066668	4027461	6684818	57		33	7495337	2883040	-1242367	4125408	6619662	27
	4	7439229	2958179	-1072519	4030698	6682655	56		34	7497262	2880472	-1248230	4128702	6617482	26
	5	7441173	2955567	-1078370	4033937	6680490	55		35	7499187	2877905	-1254094	4131999	6615300	25
	6	7443115	2952956	-1084222	4037178	6678326	54		36	7501111	2875340	-1259958	4135297	6613119	24
	7	7445058	2950347	-1090074	4040421	6676160	53		37	7503034	2872776	-1265822	4138598	6610936	23
	8	7446999	2947739	-1095927	4043666	6673994	52		38	7504957	2870214	-1271686	4141900	6608754	22
	9	7448941	2945133	-1101780	4046913	6671828	51		39	7506879	2867653	-1277551	4145205	6606570	21
	10	7450881	2942528	-1107633	4050161	6669661	50		40	7508800	2865094	-1283417	4148511	6604386	20
	11	7452821	2939925	-1113487	4053412	6667493	49		41	7510721	2862536	-1289283	4151819	6602202	19
	12	7454760	2937323	-1119341	4056664	6665325	48		42	7512641	2859980	-1295149	4155129	6600017	18
	13	7456699	2934723	-1125195	4059919	6663156	47		43	7514561	2857425	-1301016	4158441	6597831	17
	14	7458636	2932125	-1131050	4063175	6660987	46		44	7516480	2854872	-1306884	4161755	6595645	16
	15	7460574	2929528	-1136905	4066433	6658817	45		45	7518398	2852320	-1312751	4165071	6593458	15
	16	7462510	2926932	-1142761	4069693	6656646	44		46	7520316	2849770	-1318619	4168389	6591271	14
	17	7464446	2924338	-1148617	4072955	6654475	43		47	7522233	2847221	-1324488	4171709	6589083	13
	18	7466382	2921746	-1154473	4076219	6652304	42		48	7524149	2844674	-1330357	4175031	6586895	12
	19	7468317	2919155	-1160330	4079485	6650131	41		49	7526065	2842128	-1336227	4178355	6584706	11
	20	7470251	2916565	-1166187	4082753	6647959	40		50	7527980	2839584	-1342097	4181680	6582516	10
	21	7472184	2913977	-1172045	4086022	6645785	39		51	7529894	2837041	-1347967	4185008	6580326	9
	22	7474117	2911391	-1177903	4089294	6643612	38		52	7531808	2834499	-1353838	4188338	6578135	8
	23	7476049	2908806	-1183761	4092567	6641437	37		53	7533721	2831960	-1359710	4191669	6575944	7
	24	7477981	2906223	-1189620	4095843	6639262	36		54	7535634	2829421	-1365581	4195003	6573752	6
	25	7479912	2903641	-1195479	4099120	6637087	35		55	7537546	2826885	-1371454	4198338	6571560	5
	26	7481842	2901060	-1201339	4102399	6634910	34		56	7539457	2824349	-1377327	4201676	6569367	4
	27	7483772	2898482	-1207199	4105680	6632734	33		57	7541368	2821815	-1383200	4205015	6567174	3
	28	7485701	2895904	-1213059	4108963	6630557	32		58	7543278	2819283	-1389074	4208356	6564980	2
	29	7487629	2893328	-1218920	4112248	6628379	31		59	7545187	2816752	-1394948	4211700	6562785	1
	30	7489557	2890754	-1224781	4115535	6626200	30		60	7547096	2814223	-1400822	4215045	6560590	0
													Degr	41	Min

Table 6.1.1. Table page generated with Excel, for an angle of 48°. The example concerns the bottom line of the left half of this table page.

Of course, Napier didn't have access to the Excel table mentioned above. How did he work? Napier knew $\text{Sin}(48°30') = 7489557$. Napier went through the Radical Matrix M, and found in row 15 of column M28, as approximation of that Sinus, the geometric term $x_{15,28} = 7490786.610\,706$, the value which is closest to $\text{Sin}(48°30') = 7489557$. See Table 6.1.2.

In that Table we read, the corresponding arithmetic logarithm of the geometric term $x_{15,28}$, is $\lambda_{15,28} = 2\,889\,112.795$. With inequality (3.5.6) we find next Formula:

$$10^7 \frac{x_{15,28} - \text{Sin}\,48°30'}{x_{15,28}} < \lambda\left(\text{Sin}\,48°30'\right) - \lambda_{15,28} < 10^7 \frac{x_{15,28} - \text{Sin}\,48°30'}{\text{Sin}\,48°30'} \qquad (6.1.1)$$

	Column M28 of matrix M	
p	**Geom. Term**	**Aritm. Log.**
0	7547192.872036	2814094.039
1	7543419.275600	2819095.289
2	7539647.565963	2824096.540
3	7535877.742180	2829097.790
4	7532109.803308	2834099.041
5	7528343.748407	2839100.291
6	7524579.576533	2844101.541
7	7520817.286744	2849102.792
8	7517056.878101	2854104.042
9	7513298.349662	2859105.293
10	7509541.700487	2864106.543
11	7505786.929637	2869107.794
12	7502034.036172	2874109.044
13	7498283.019154	2879110.294
14	7494533.877644	2884111.545
15	7490786.610706	2889112.795
16	7487041.217400	2894114.046
17	7483297.696791	2899115.296
18	7479556.047943	2904116.546
19	7475816.269919	2909117.797
20	7472078.361784	2914119.047

Table 6.1.2. In column 28, row 15, of the Radical Matrix M, we find the nearest approximation of the Sinus of 48°30′ and its corresponding arithmetic logarithm.

It follows:

$$2\,890754{,}015\,146 < \lambda\left(\text{Sin}\,48°30'\right) < 2\,890754{,}284\,551 \qquad (6.1.2)$$

In seven significant digits, so 2 890 754, the arithmetic logarithm λ of $\text{Sin}\left(48°30'\right)$ is exactly the Napier logarithm *L* we find on the bottom line in Table 6.1.1, indeed.

This lengthy calculation pertains to a single Sine value. One can only imagine the tedium involved in the pen-and-paper construction of a comprehensive trigonometric logarithm table during the 17th century.

6.2. The logarithm of the Sine of major angles

If the angle θ is close to 90°, then the Sinus, $x = \text{Sin}\,\theta$ is in the proximity of the rightmost point *R* of the interval OR. In that case the calculation of the logarithm of it can be simplified considerably. For the difference $\Delta x = R - x$ the following linear approximation applies:

$$L(\text{Sin}\,\theta) = R \cdot \ln \frac{R}{\text{Sin}\,\theta} = -R \ln\left(1 - \frac{\Delta x}{R}\right) \approx -R \cdot -\frac{\Delta x}{R} = \Delta x = R - \text{Sin}\,\theta \qquad (6.2.1)$$

By applying inequality (3.5.2) we get as a result:

$$R - \mathrm{Sin}\theta < L(\mathrm{Sin}\theta) < \frac{R}{\mathrm{Sin}\theta}(R - \mathrm{Sin}\theta) \tag{6.2.2}$$

The difference between the left and right boundaries of this inequality is less than 1 for angles with $\mathrm{Sin}\theta > 999684$, hence for $90° > \theta > 84°34'$. For those major angles, without error, in seven significant digits, approximation (6.2.1) can be used. See also Havil (2014), [2], pp. 118 and 122 – 123.

6.3. The Logarithm of the Sine of small angles

For angles greater than 30° we can always proceed as above in Paragraph 6.1.
The exact logarithm for 30° is, to seven significant digits, $L(H) = 6\ 931\ 472$. See Table 3.4.1. Via the inequality in Formula (3.5.6), Napier arrived at the arithmetic approximation $\lambda(H) = 6\ 931\ 471.805\ 599$, which is again an astonishingly accurate estimate. See Havil (2004), [2], p. 120. And see Paragraph 5.8 of this essay.

Geometric numbers *x* to the left of point H on the upper number line of the kinematic model, can easily be converted to a value right of H. For that purpose, Napier used the doubling Formula or the decade Formula, if necessary, in combination.

6.3.1. The doubling Formula

The doubling Formula is easy to understand. The term 2*x* is greater than *x*; the logarithm of 2x must be smaller than the logarithm of *x*. From the equality of two ratios, we find the *doubling Formula* for the arithmetic logarithms:

$$\begin{cases} \lambda\left(\dfrac{2x}{x}\right) = \lambda\left(\dfrac{R}{H}\right) \rightarrow \lambda(x) = \lambda(2x) + \lambda(H) \\ \lambda(H) = 6\ 931\ 471.805\ 599 \end{cases} \tag{6.3.1}$$

By advancing to the right by a factor of 2 on the upper number axis of the kinematic model, we can determine the logarithm of a smaller value *x* using the logarithm of the larger 2*x*. Repeating this process allows us to jump a total factor of 4 to the right, and this pattern continues. In general, we derive:

$$\lambda(x) = \lambda(2^n x) + n\lambda(H) \tag{6.3.2}$$

By repeatedly doubling, we can transform a value *x* located to the left of point H into a value that lies to the right of H. This new value can be approximated by a geometric term found in one of the columns of the Radical Matrix M. We can then calculate the corresponding arithmetic logarithm in the same manner as demonstrated in the example in Paragraph 6.1.1. above.

6.3.2. The decade Formula

From Formula (3.2.6), $L(x) = R \cdot \ln\frac{R}{x}$, we deduce:

$$\begin{cases} \Delta = L(1) - L(10) = R\ln\left(\frac{R}{1}\right) - R\ln\left(\frac{R}{10}\right) = R\ln 10 \\ \Delta = 23025850.92994050 \approx 2.3 \cdot 10^7 \end{cases} \quad (6.3.3)$$

Because $L\left(10^n x\right) = R\ln\left(\frac{R}{x}\cdot\frac{1}{10^n}\right) = R\ln\left(\frac{R}{x}\right) + R\ln\left(\frac{1}{10^n}\right)$, the *decade Formula* follows:

$$L(x) = L\left(10^n x\right) + n\Delta \quad (6.3.4)$$

6.3.3. Combining the two Formulas

For example, by combining Formulas (6.3.2) and (6.3.4), we can make a jump to the right half of interval OR, from x to $20x$ and calculate the logarithm of x via the transformed value $20x$:

$$\begin{cases} \lambda(x) = \lambda\left(20x\right) + \lambda(H) + \Delta \\ \lambda(H) + \Delta = 6\,931\,471.805\,599 + 23025850.92994050 = 29957322.74 \end{cases} \quad (6.3.5)$$

Napier even gives a Table in the *Constructio* with values of the difference logarithms for common jumps $2^m 10^n$:

$$\lambda(x) - \lambda\left(2^m 10^n x\right) = m\lambda(H) + n\Delta \quad (6.3.6)$$

See the Short Table in Macdonald (1889), [1], p. 39, and in Havil (2004), [2], p. 122.

6.3.4. Example

As an example, we calculate the logarithm of the Sine of the relatively small angle 14°41'. We apply Formula (6.3.1). We know the x-value of the Sine: 2534765.670689. This is a value left of H. We double this, and get 5069531.341378, that is a value just right of H.

x = Sin(14°41') =	2534765.670689						
$2x$ =	5069531.341378						
$x_1 = x_{67,11}$ =	5071878.264413		λ_1 =	6788738.776102			
$x_2 = x_{67,12}$ =	5069342.325280		λ_2 =	6793740.026518			
						$\lambda(H)$ =	6931471.8
			Lin Int. =	6793367.259		$\lambda(x)$ =	13724839

Table 6.3.1. Calculation of the logarithm of the Sine of 14°41' to 8 significant figures.

In column 67 of the Radical Matrix M, we find in entries 11 and 12 two x-values between which $2x$ lies. In Radical Matrix M we also find their arithmetic logarithms λ_1 and λ_2.

The linear interpolation of these two logarithms gives the logarithm 6793367.259 .We add $\lambda(H) = 6\,931\,471.805\,599$ and ultimately, we find the arithmetic logarithm of the Sin(14°41'), which is 13 724 839 in no less than eight significant figures. Notice, it even equals the exact logarithm! See Table 6.3.1.

CHAPTER 7
FINAL REMARKS AND CONCLUSIONS

7.1. A geometric and an arithmetic sequence

Drawing inspiration from his exploration of a continuous kinematic model, Napier discovered a foundational discretization of the real numbers within a specified interval $OR = \left(0,10^7\right)$, along with their corresponding arithmetic logarithms:

$$x_n = R\left(1-\frac{1}{R}\right)^n \leftrightarrow \lambda(x_n) \approx n\left(1+\frac{1}{2R}\right) \quad (7.1.1)$$

The geometric sequence of terms $x_n = R\left(1-\frac{1}{R}\right)^n$ in Formula (7.1.1) is a discretization of the interval OR on the upper number axis of Napier's model; the corresponding arithmetic sequence with logarithmic terms $\lambda(x_n) \approx n\left(1+\frac{1}{2R}\right)$ is a discretization of the lower positive number axis.

The difficult arithmetic, with the Sinuses x, with values between 10^7 and 0, that are approximated by terms of the geometric sequence $\{x_n\}$, is therefore converted into easier arithmetic with the logarithms, that are approximated by terms of the arithmetic sequence $\{\lambda_n\}$.

To this end of approximation, Napier constructed a 21 x 69 Radical Matrix M, with selected double-geometric terms and associated arithmetic logarithms:

$$x_{p,q} = R\left(1-\frac{1}{2000}\right)^p\left(1-\frac{1}{100}\right)^q;\ p=0,1,2,3,\ldots,20;\ q=0,1,2,3,\ldots,68 \quad (7.1.2)$$

To determine the two starting points of the Radical Matrix M, he designed two Reference Tables, we call T1 and T2. With reference to the Radical Matrix and a small number of now well-known mathematical Formulas, but developed by himself, Napier was able to draw up a logarithm table for trigonometrical quantities for the 5,401 angles in minutes, in the first quadrant of a circle.

7.2. The mechanization of worldview

Napier's work distinctly highlights his significant role in the mechanization of the worldview — a crucial transition from classical-medieval natural science to a more analytical and mathematical comprehension of physical reality. Notably, the concept of instantaneous velocity, which posed a philosophical dilemma for many around the year 1600, appears to have been intuitively understood by Napier. His kinematic model of thought hinges on this concept, demonstrating his advanced grasp of motion and dynamics.

In his work on logarithms, one can also see some numerical approximations based on the Maclaurin series for the functions $f(x) = (1+x)^\alpha$ and $f(x) = \ln(1+x)$, that were found only a hundred years later. For example, we formulate the inequality in Formula (3.5.1),

$X > 1 \rightarrow \frac{X-1}{X} < \ln X < X-1$, in terms of the natural logarithm nowadays, but we see that Napier already understood that inequality without the modern formulation.

Undoubtedly, it can be argued that Napier was on the verge of uncovering the natural logarithm and Euler's constant *e*. The precise and continuous logarithm developed from our continuous kinematic model serves as a modern logarithm with a base of 1/*e* albeit adjusted by a multiplicative factor $R = 10^7$.

In the St. Andrews MacTutor biography of Napier [*3], we find a description that serves as a fitting conclusion:

> *"Napier will be remembered for making one of the most important contributions to the advance of knowledge. It was through the use of logarithms that Kepler was able to reduce his observations and make his breakthrough which then in turn underpinned Newton's theory of gravitation. In the preface to the Mirifici logarithmorum canonis descriptio, Napier says he hoped that his logarithms will save calculators much time and free them from the slippery errors of calculations. Laplace, 200 years later, agreed, saying that logarithms…*
> *...by shortening the labours, doubled the life of the astronomer".*

7.3. Conclusions

- Logarithms were invented independently by John Napier and Jost Bürgi.
- Their logarithms differed from each other and from the ordinary and natural logarithms now in use.
- The Napier logarithms were published in 1614; Bürgi's logarithms were published in 1620.
- Neither had the concept of a logarithmic base.
- Napier mathematics was way ahead of his time.
- Napier is re-inventor of the decimal point and the 'modern' decimal notation of fractions.
- Napier defined logarithms as a ratio of two distances in a geometric form, as opposed to the current definition of logarithms as exponents.
- Actually, he already used analytical mathematics founded more than a century later, but obviously not in the formulation we use today.
- Napier used smart interpolations that were far later founded.
- The logarithm of Napier is not a logarithm with a specific base in a modern sense.
- The possibility of defining logarithms as exponents was discovered by John Wallis in 1685 and by Johann Bernoulli in 1694; later synthesis by Euler (c. 1730).
- The logarithm tables (for instance the famous one by Adriaan Vlacq and Ezechiël de Decker, published during the years 1626-1628) are based on the joint effort of Napier and Henry Briggs (1624).
- The logarithm of Napier is NOT the natural logarithm.
- The natural logarithm arose as a more or less coincidental variation on the Napier logarithm.
- Napier took 20 years to do the calculations for construction of his table.

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